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Classical Locally Finite Tits Chamber Systems of Rank 3

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1. INTRODUCTION

A *chamber system* $\mathscr{C} = (\mathscr{C}, (\not{p}_i)_{i \in I})$ over some index set I is a set $\mathscr{C}$ of "chambers" together with partitions $\not{p}_i$, $i \in I$ of $\mathscr{C}$. If $J \subseteq I$ and $c \in \mathscr{C}$ then $\not{p}_J$ is the join of all partitions $\not{p}_j$, $j \in J$ and $\Delta_J(c)$ is the element of $\not{p}_J$ containing c. $\Delta_J(c)$ is again a chamber system over J with $\not{p}_j$ restricted to J. We say $\mathscr{C}$ is *connected* if $\Delta_I(c) = \mathscr{C}$. Furthermore $|I|$ is the *rank* of $\mathscr{C}$.

The most important examples of chamber systems are those obtained from groups. If G is a group, B a subgroup of G, and X_i, $i \in I$ are subgroups of G containing B then $\mathscr{C} = \mathscr{C}(G; B; (X_i)_{i \in I})$ has as chambers the cosets Bg, $g \in G$, two such cosets Bg, Bh being i-adjacent if and only if $X_i g = X_i h$. Then G acts (chamber)-transitively on $\mathscr{C}$ with kernel $B_G = \bigcap B^g$ and $\mathscr{C}$ is connected if and only if $G = \langle X_i \mid i \in I \rangle$. Furthermore, each chamber system with transitive automorphism group G can be obtained in this way by setting $B = G_c$ and $X_i = G_{\Delta_i(c)}$.

A rank 2 chamber system $\mathscr{C} = (\mathscr{C}, \not{p}_i, \not{p}_j)$ is called a *classical generalized* m_{ij}-gon ($m_{ij} \geqslant 3$) if $\mathscr{C}$ is isomorphic to $\mathscr{C}(G; B; P_i, P_j)$, where G is an (essentially) simple rank 2 group of Lie type (in the sense of [1]), B is a Borel subgroup, and P_i, P_j are the two maximal parabolic subgroups of G containing B. Here m_{ij} is given by the relation $w_i^2 = w_j^2 = (w_i w_j)^{m_{ij}} = 1$ defined by the Weyl group of G. A rank 2 chamber system is a *generalized digon* if $\mathscr{C}$ is isomorphic to $\mathscr{C}(G, P_i \cap P_j, P_i, P_j)$, where P_i, P_j are different proper subgroups of G satisfying $G = P_i P_j = P_j P_i$. A connected chamber system $\mathscr{C}$ over I is called a *classical Tits chamber system* if for each $c \in \mathscr{C}$ and $\{i, j\} \subseteq I$, $\Delta_{i,j}(c)$ is either a generalized digon or a classical generalized m_{ij}-gon (for some $m_{ij} \in \mathbb{N}$). $\mathscr{C}$ is *locally finite* if all $\Delta_{i,j}(c)$ are finite. For motivation to consider such classical locally finite Tits chamber systems see [15] and [16].

If $\mathscr{C}$ is a classical Tits chamber system then the diagram $\Delta(I)$ is defined in

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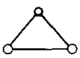
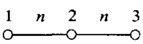
the usual way (i.e., by connecting the nodes i and j by a bond of strength $m_{ij}-2$!)

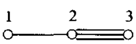
In [17] we have determined all such chamber systems with transitive automorphism group and finite chamber stabilizer which are defined over "large" fields. In this paper we treat the case which is probably the most important for finite group theory purposes, namely the case where rank $\mathscr{C}=3$. More precisely, we prove:

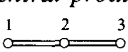
THEOREM 1. *Suppose $\mathscr{C}$ is a classical, locally finite Tits chamber system of rank 3 with connected diagram Δ and transitive automorphism group G with finite chamber stabilizer $B=G_c$. For $i\in I=\{1,2,3\}$ let $G_i=G_{\Delta_{I-i}(c)}$ and K_i be the kernel of the action of G_i on $\Delta_{I-i}(c)$. Then one of the following holds*:

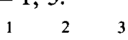
(1) *$\mathscr{C}$ is either a finite spherical building or the chamber system of the A_7-geometry of type C_3. Further, either G is an extension of a finite group of Lie type, Δ, by diagonal and field automorphisms or $G\simeq A_7$.*

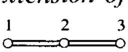
(2) $\Delta=$  *and either $G_i\simeq F_{21}$ for each $i\in I$ or $G_i\simeq F_{73\cdot 9}$ for each $i\in I$.*

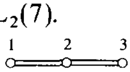
(3) $\Delta=$ , *$n=4$, 6, or 8 and $G_1\simeq G_3$ is an extension of a rank 2 Lie type group defined over* GF(2) *or* GF(3) *by diagonal and field automorphisms. Further G_2 is solvable.* (GF(2) *resp.* GF(3) *is the "fixed field" in case G_1 is twisted. Moreover, field automorphisms can only occur in the twisted case.*)

(4) $\Delta=$ , *$G_1\simeq G_2(p)$, $G_3\simeq(\mathbb{Z}_p^3)\mathrm{SL}_3(p)$ (non-split), $p=2$ or 5. Further $G_2\simeq(Q_8*Q_8)(\mathbb{Z}_3\times\mathbb{Z}_3)\mathbb{Z}_2$ in case $p=2$ resp. $G_2\simeq 5^{1+4}(\hat{\Sigma}_6*\mathbb{Z}_4)$ in case $p=5$. (Here $\hat{\Sigma}_6$ is a perfect central 2-covering of Σ_6 and $*$ means central product.)*

(5) $\Delta=$ , *$2^4\leqslant|K_i|\leqslant 2^5$ for $i=1,3$. $|K_1\cap K_3|=4$ and K_i is an indecomposable module for $G_i/K_i\in\{\Sigma_6,A_6\}$ with $C_{K_i}(G_i)=1$. Moreover, either the extensions of G_i/K_i by K_i split for $i=1,3$ or $G_i/K_i\simeq\Sigma_6$ and $|K_i|=2^4$ for $i=1,3$.*

(6) $\Delta=$ , *$G_1\simeq 2^{1+6}\Omega^-(6,2)$ (resp. $O^-(6,2)$), $G_3=2^{4+6}\hat{A}_6$ (resp. $\hat{\Sigma}_6$), and $G_2=2^{2+8}(\Sigma_3\times A_5)$ (resp. $\Sigma_3\times\Sigma_5$). (Here $\hat{A}_6$ resp. $\hat{\Sigma}_6$ is the perfect extension of A_6 resp. Σ_6* by $\mathbb{Z}_3$.)

(7) $\Delta=$ , *$G_1\simeq G_3\simeq P\mathrm{Sp}(4,7)$, and $O^{7'}(G_2)\simeq(7^{1+2}\times\mathbb{Z}_2^3)$ $\mathrm{SL}_2(7)$.*

(8) $\Delta=$ , *$K_1=K_3=1$, and $G_i\simeq(\mathbb{Z}_3\times P\mathrm{Sp}(4,3))\mathbb{Z}_2$ for $i=1,3$.*

(9) *There exists a* $j \in I$ *such that* $\Delta_{I-j}(c)$ *is the* $\mathrm{Sp}(4,3)$ *or* $U_4(3)$ *generalized quadrangle. Further* $G_j/K_j \in \{2^4F_{20},\ L_3(4)\cdot 2,\ 2^4A_5\}$.[1]

The exact structure of G_2 in cases (3) and (8) is easily determined, since $G_2 = P_1P_3 = P_3P_1$ with $P_i = G_{\Delta_i(c)}$. Examples of nearly all possible types in (2)–(8) either are well known or have been constructed in [8]. The proof of Theorem 1 depends on Theorem 2 of [17], where the determination of such chamber systems has been reduced to the determination of the corresponding parabolic systems. To define those we need some further notation.

Let $\mathscr{L}_p^i$ be the set of finite simple rank i-groups of Lie type in char p (in the sense of [1]) together with:

$\mathrm{PSL}_2(3)$ and ${}^2G_2(3)$ if $i=1$, $p=3$,

$\mathrm{PSL}_2(2)$, $\mathrm{PSU}_3(2)$, and $\mathrm{Sz}(2) \simeq F_{20}$ if $p=2$ and $i=1$,

A_6, Σ_6, $G_2(2)'$, $G_2(2)$, ${}^2F_4(2)'$, and ${}^2F_4(2)$ if $p=2$ and $i=2$.

A system $\not{p} = \{X_i \mid i \in I\}$ of pairwise different subgroups generating the group G is called a *parabolic system* of G of char p, if it satisfies:

(1) There exists a finite p-subgroup $S \leqslant \bigcap_{i \in I} X_i$ such that $S \in \mathrm{Syl}_p(X_{i,j})$, $X_{i,j} = \langle X_i, X_j\rangle$ for all $i, j \in I$.

(2) $\bar{X}_i = O^{p'}(X_i/O_p(X_i))$ is a perfect central extension of a group in $\mathscr{L}_p^1$.

(3) Either $\bar{X}_{i,j}$ is a perfect central extension of a group in $\mathscr{L}_p^2$ or

$$(\bar{Y}_i * \bar{Y}_j)\bar{S} \subseteq \bar{X}_{i,j} \overset{\sim}{\subset} \bar{X}_i * \bar{X}_j,$$

where $\bar{Y}_i = O^p(\bar{X}_i)$ and $*$ means central product. Further, if $\bar{X}_{i,j} \simeq \hat{A}_6$, then $N_{O^{2'}(X_{i,j})}(S) \leqslant X_i \cap X_j$.

(A perfect central extension does not need to be perfect itself. The extra condition in (3) is only needed in case of $\hat{A}_6$. In the other cases we can force this by slightly enlarging the groups X_i. See [15, (4.1)].)

We call $\not{p}$ a quasiparabolic system of G, if we also allow in (3) the possibility that $\bar{X}_{i,j} \simeq \hat{\Sigma}_6$ or $\hat{A}_6$ and $\bar{X}_i \simeq \bar{X}_j \simeq \Sigma_3$. The diagram $\Delta = \Delta(\not{p})$ is defined in the obvious way, where we write $\overset{i}{\circ}\overset{\sim}{=\!=}\overset{j}{\circ}$ if and only if $\bar{X}_{i,j} \simeq \hat{\Sigma}_6$ or $\hat{A}_6$ but $N_{O^{2'}(X_{i,j})}(S) \nleqslant X_i \cap X_j$.

Now we can state:

THEOREM 2. *Suppose* $\not{p}$ *is a quasiparabolic system of rank* 3 *and* char. p *of the group* G *with connected diagram* Δ. *Let* $G_0 = \langle O^{p'}(X_i) \mid i \in I\rangle$,

[1] Thomas Meixner has shown in [9] that $\Delta = \overset{i}{\circ}=\!=\circ=\!=\overset{j}{\circ}$, $K_i = K_j = 1$, and $G_i \simeq G_j$ in this case.

$I=\{1, 2, 3\}$, $G_i=\langle O^{p'}(X_j) \mid j\in I-i\rangle$, $N=S_{G_0}=\bigcap_{g\in G_0} S^g$, $\bar{G}_0=G_0/N$, and $\bar{G}_i=G_i/N$, where S is the common p-Sylow subgroup of each X_i. Then up to symmetry one of the following holds:

(1) *Δ is spherical and $\bar{G}_0$ is a perfect central extension of a finite group of Lie type Δ in* char p, *or $\Delta=A_3$ or C_3 and $\bar{G}_0\simeq A_7$.*

(2) $\Delta=\overset{1}{\circ}\overset{n}{\text{———}}\overset{2}{\circ}\overset{n}{\text{———}}\overset{3}{\circ}$, *$n=4$, 6, or 8, and $\bar{G}_1\simeq\bar{G}_3$ is a rank 2 Lie type group over* GF(2) *or* GF(3) *and $\bar{G}_2$ is solvable.*

(3) $\Delta=\overset{1}{\circ}\overset{}{\text{———}}\overset{2}{\circ}\overset{}{\equiv\!\equiv\!\equiv}\overset{3}{\circ}$, $\bar{G}_1\simeq G_2(2)$, $\bar{G}_3\simeq\mathbb{Z}_2^3L_3(2)$ (non-split), and $\bar{G}_2\simeq(Q_8 * Q_8)(\mathbb{Z}_3\times\mathbb{Z}_3)\mathbb{Z}_2$.

(4) $\Delta=\overset{1}{\circ}\overset{}{=\!=\!=}\overset{2}{\circ}\overset{}{=\!=\!=}\overset{3}{\circ}$, $\overset{1}{\circ}\overset{\sim}{=\!=\!=}\overset{2}{\circ}\overset{\sim}{=\!=\!=}\overset{3}{\circ}$, *or* $\overset{1}{\circ}\overset{}{=\!=\!=}\overset{2}{\circ}\overset{\sim}{=\!=\!=}\overset{3}{\circ}$. *$\bar{G}_i/O_2(\bar{G}_i)\in\{A_6, \Sigma_6, \hat{A}_6, \hat{\Sigma}_6\}$ and $2^4\leqslant|O_2(\bar{G}_i)|\leqslant 2^5$ for $i=1$, 3. Further, $O_2(\bar{G}_i)$ is an indecomposable $\bar{G}_i$-module with $C_{O_2(\bar{G}_i)}(\bar{G}_i)=1$ and the extension of $\bar{G}_i/O_2(\bar{G}_i)$ by $O_2(\bar{G}_i)$ splits except in the case $\bar{G}_i/O_2(\bar{G}_i)\simeq\Sigma_6$ and $|O_2(\bar{G}_i)|=2^4$ for $i=1$ and* 3.

(5) $\Delta=\overset{1}{\circ}\overset{}{=\!=\!=}\overset{2}{\circ}\overset{}{=\!=\!=}\overset{3}{\circ}$ *with $\bar{G}_1\simeq 2^{1+6}\Omega^-(6, 2)$ (non-split), $\bar{G}_3=2^{4+6}\hat{A}_6$, and $\bar{G}_2\simeq 2^{2+8}(\Sigma_3\times A_5)$.*

(6) $\Delta=\overset{1}{\circ}\overset{}{\text{———}}\overset{2}{\circ}\overset{\sim}{=\!=\!=}\overset{3}{\circ}$, $\bar{G}_1=2^6\hat{\Sigma}_6$, $\bar{G}_3=2^{1+6}L_3(2)$.

(7) $\Delta=\overset{1}{\circ}\overset{}{\text{———}}\overset{2}{\circ}\overset{\sim}{=\!=\!=}\overset{3}{\circ}$, *$\bar{G}_1=2^5\hat{\Sigma}_6$ (where 2^5 is the orthogonal* Sp(4, 2)*-module), $\bar{G}_3=2^6L_3(2)$ (where $O_2(\bar{G}_3)$ is an indecomposable $\bar{G}_3$-module).*

(8) $\Delta=\overset{1}{\circ}\overset{}{=\!=\!=}\overset{2}{\circ}\overset{\sim}{=\!=\!=}\overset{3}{\circ}$ *or* $\overset{1}{\circ}\overset{\sim}{=\!=\!=}\overset{2}{\circ}\overset{\sim}{=\!=\!=}\overset{3}{\circ}$ *and* $\{\bar{G}_1, \bar{G}_3\}\subseteq\{A_6, \Sigma_6, \hat{A}_6, \hat{\Sigma}_6\}$.

(9) $\Delta=\overset{1}{\circ}\overset{}{=\!=\!=}\overset{2}{\circ}\overset{}{=\!=\!=}\overset{3}{\circ}$, $\bar{G}_1\simeq\bar{G}_3\simeq(\mathbb{Z}_3\times P\mathrm{Sp}(4, 3))\mathbb{Z}_2$.

(10) $\Delta=\overset{1}{\circ}\overset{}{\text{———}}\overset{2}{\circ}\overset{\sim}{=\!=\!=}\overset{3}{\circ}$, *$\bar{G}_1\leqslant N(\bar{Z})$, where $\bar{Z}=\Omega_1(Z(\bar{S}))\simeq\mathbb{Z}_2$, $\bar{G}_1/O_2(\bar{G}_1)\simeq\hat{\Sigma}_6$, and $\langle\bar{Z}^{\bar{G}_3}\rangle$ is a natural $L_3(2)$-module for $\bar{G}_3/O_2(\bar{G}_3)$.*

It has been shown by P. Rowley in [12] that case (10) does not occur. But our general argument for the proof of Theorem 3 below breaks down in this special case. So the treatment of case (10) would need extra amalgam-type arguments, which we, to save space, do not include here. The structure of the "middle group" is easily deduced from the definition of quasiparabolic systems in those cases, where it has not been stated explicitly.

To prove Theorem 2 one may assume that $G=G_0$, $N=1$, and, by the main theorem of [19], Δ is linear. Moreover, by [17, (3.2)], case (1) holds if Δ is spherical. If now $Z=\Omega_1(Z(S)) \ntrianglelefteq G_i$, $i=1$ and 3 (where 1 and 3 are the end nodes of Δ), then Theorem 2 is a special case of [14]. (We only need to discuss in Section 7 which of the cases of [14] gives rise to a quasiparabolic system.) So the proof of Theorem 2 actually reduces to the proof of:

THEOREM 3. *Suppose $\not{p}$ is a rank 3 quasiparabolic system of the group G with connected, non-spherical, linear diagram Δ. Suppose in addition that the following hold:*

(a) $G = \langle O^{p'}(X_i) \mid i \in I = \{1, 2, 3\} \rangle$ *and if* $B_i = N_{O^{p'}(X_i)}(S)$, $B = \langle B_i \mid i \in I \rangle$, *then* $B_G = 1$.

(b) $C_{\tilde{G}_i}(M_i) \leqslant M_i$ *for each* $i \in I$; *where* G_i *is as in Theorem* 2, $\tilde{G}_i = O^{p'}(G_i)$, $M_i = S_{G_i}$.

(c) $Z = \Omega_1(Z(S)) \lhd G_j$ *for some end node* j *of* Δ.

Then, up to symmetry, one of the following holds:

(1) $\Delta = \overset{1}{\circ}\!=\!=\!\overset{2}{\circ}\!=\!=\!\overset{3}{\circ}$, $G_1 \simeq 2^{1+6}\Omega^-(6, 2)$ *(non-split)*, $G_3 \simeq 2^{4+6}\hat{A}_6$, *and* $G_2 = 2^{2+8}(\Sigma_3 \times A_5)$.

(2) $\Delta = \overset{1}{\circ}\!-\!-\!\overset{2}{\circ}\overset{\sim}{=\!=}\overset{3}{\circ}$, $G_1 \simeq 2^6\hat{\Sigma}_6$, $G_3 \simeq 2^{1+6}L_3(2)$, *and* $G_2 \simeq 2^{2+6}(\Sigma_3 \times \Sigma_3)$. (*For the exact structure of* G_2 *in* (1) *and* (2), *see* (6.2), (6.4).)

(3) $\Delta = \overset{1}{\circ}\!-\!-\!\overset{2}{\circ}\overset{\sim}{=\!=}\overset{3}{\circ}$, $G_1 \leqslant N(Z)$, $Z \simeq \mathbb{Z}_2$, *and* $\langle Z^{G_3} \rangle$ *is the natural* $L_3(2)$*-module for* $G_3/O_2(G_3)$.

(As mentioned above, case (3) does not occur by a result of P. Rowley.)

The proof of Theorem 3 is very technical. It depends on a theorem of Niles [10], which essentially shows that we may assume that $\not{p}$ is defined over GF(2) or GF(3) (see (2.2)–(2.4)) on certain "amalgam-type" arguments and on special properties of GF(p)-representations of the rank 1 and 2 Lie-type groups.

Finally a comment on the hypothesis of Theorem 2. It seems possible that one can prove in the future results similar to those in [19] and [14] just using the "local" structure of the $\bar{G}_i$ provided by [6]. But since this may take some time and since the higher rank classifications in char 2 by G. Stroth and in char 3 by Th. Meixner depend on the so far unpublished treatment of the rank 3 case, it still seems worthwhile to publish this result.

2. NOTATION AND PRELIMINARY RESULTS

(2.1) *Notation.* Classical Tits chamber systems, parabolic and quasiparabolic systems have been defined already in the introduction. So we do not repeat the definitions here.

If now $\not{p} = \{X_i \mid i \in I\}$ is a parabolic system of characteristic p with common p-Sylow subgroup S of the group G, we use the following reduction: Let $\tilde{X}_i = O^{p'}(X_i)$, $B_i = N_{\tilde{X}_i}(S)$, $B = \langle B_i \mid i \in I \rangle$, and $P_i = \tilde{X}_i B$.

Then by [15, (4.1)] $S \trianglelefteq B$ and B/S is a p'-group. Further $B \leqslant N(\tilde{X}_i)$, $\{P_i \mid i \in I\}$ is again a parabolic system of $G_0 = \langle \tilde{X}_i \mid i \in I \rangle \trianglelefteq G$ of the same type, and

$$B = N_{P_i}(S) = N_{P_{ij}}(S) \qquad \text{for all } i, j \in I,$$

where $P_{ij} = \langle P_i, P_j \rangle$. So when we consider parabolic systems[2] in Sections 2–6 we always assume $P_i = X_i$. We call B the *Borel subgroup* of the system $\not{p}$ and fix a p'-complement H to S in B, which will sometimes be called the Cartan subgroup. By definition of B we have

$$H = \prod_{i \in I} H_i, \qquad \text{where} \quad H_i = H \cap O^{p'}(P_i).$$

Hence each H_i is cyclic. Since $[H_i, H_j] = 1$ by the structure of $\tilde{P}_{ij} = O^{p'}(P_{ij})$ we find that H is an abelian p'-group of rank at most $n = |I|$.

Since this paper deals only with parabolic and quasiparabolic systems of rank 3 we use the folowing further notation (if $|I| = 3$!):

$G_k = P_{ij}$ if $I = \{i, j, k\}$
$Q_i = S_{P_i}$
$K_i = B_{G_i}$
$M_i = S_{G_i}$
$\tilde{P}_i = O^{p'}(P_i)$
$\tilde{G}_i = O^{p'}(G_i)$
$\bar{P}_i = \tilde{P}_i/Q_i$, and
$\bar{G}_i = \tilde{G}_i/M_i$.

Then by definition of $\not{p}$, $\bar{G}_i$ is a perfect central extension of a rank 2 Lie-type group if $\Delta(i)$ is connected, resp. $\bar{G}_i \underset{\sim}{\simeq} \hat{A}_6$ or $\hat{\Sigma}_6$ and $P_j \simeq \tilde{G}_i/Q_j \simeq P_k \cap \tilde{G}_i/Q_k \simeq \Sigma_3$. In the latter case $\Delta(i)$: $\circ\!\!=\!\!=\!\!\circ$.

Any connected rank 3 Coxeter diagram, which is not spherical, will be called hyperbolic. Using a theorem of Niles [10] one has the following:

(2.2) LEMMA. *Suppose $\not{p}$ is a parabolic system of rank* 3 *with hyperbolic linear diagram Δ of the group G. Then the following hold*:

(1) *Char* $\not{p} = 2$ *or* 3

(2) *One $\bar{P}_i$ is defined over* GF(2) *or* GF(3) *or* $\Delta = \underset{0}{\circ}\!\!-\!\!\!-\!\!\underset{1}{\circ}\!\!=\!\!=\!\!\underset{2}{\circ}$ *with* $\bar{G}_2 \simeq (P)\,\mathrm{SL}_3(4)$ *but* $(P)\,\mathrm{GL}_3(4) \ntrianglelefteq G_2/M_2$.

Proof. Suppose (2.2) does not hold. Then the proof of Theorem B of Niles [10] shows that $\tilde{G}\langle \tilde{P}_i \mid i \in I \rangle$ has a BN pair of type Δ with Borel

[2] The parabolic systems of this paper are the weak parabolic systems of [15].

subgroup B. But as Δ is non-spherical [17, (2.7)] implies that B is infinite, a contradiction to the definition of a parabolic system.

The next lemma is an elementary observation:

(2.3) LEMMA. *Suppose* $\mathfrak{p} = \{P_i \mid i \in I\}$ *is an arbitrary parabolic system (with the conventions of* (2.1)) *satisfying* $B_G = 1$; *then* $C_H(\bar{G}_i)$ *is cyclic for each* $i \in I$.

Proof. Suppose (2.3) is false for some i. As $\bar{P}_i$ is a rank 1 Lie-type group, H can only induce a cyclic diagonal group on $\bar{P}_i$. (If H would induce p'-field automorphisms on $\bar{P}_i$, then some H_j must induce field automorphisms on $\bar{P}_i$, a contradiction to the structure of $\bar{P}_{i,j}$!) Hence $1 \neq H_0 = C_H(\bar{G}_i) \cap C_H(\bar{P}_i)$ and

$$[H_0, S] = [H_0, \tilde{P}_i] = [H_0, \tilde{G}_i] \lhd G = \langle \tilde{G}_i, \tilde{P}_i \rangle.$$

As $B_G = 1$ this implies $[H_0, S] = 1$ and thus

$$H_0 \leqslant C_B(\tilde{G}_i) \cap C_B(\tilde{P}_i) \leqslant C_B(G) = 1,$$

a contradiction.

(2.4) LEMMA. *Suppose G is either a (simple) classical group of Lie rank* 2 *over* GF(3) *or* ${}^3D_4(3)$ *resp.* ${}^3D_4(2)$. *Let* $S \in \mathrm{Syl}_p(G)$ *where* $p = 3$ *(resp.* $p = 2$ *in the case of* ${}^3D_4(2)$), $B = N_G(S)$, P_1 *and* P_2 *the two maximal parabolics of G containing B with unipotent radicals* U_1, U_2, *and H a p'-complement to S in B. Then the following hold*:

(1) $\Phi(S) = U_1 \cap U_2$

(2) $U_i/\Phi(S)$, $i = 1, 2$ *are the only H-invariant proper subgroups of* $S/\Phi(S)$.

Proof. In the case of a classical group over GF(3) (2.4) is a lemma of Th. Meixner, which will appear in [9, (3.1)]. (But the proof is an easy exercise, which the reader might want to solve on his own!) So assume $G \simeq {}^3D_4(p)$, $p = 2$ or 3. Suppose $P_1 = U_1 L_1$ is the parabolic with $L_1 \simeq L_2(8)$ resp. $L_2(3^3)$. Then U_1 is extraspecial of order p^{1+8} and $U_1/2(U_1)$ is the irreducible $\mathrm{GF}(p)L_1$-module corresponding to the $\mathrm{GF}(p^3)L_1$-module $N \otimes N\sigma \times N\sigma^2$, where N is the natural $\mathrm{GF}(p^3)L_1$-module and $\langle \sigma \rangle = \mathrm{Gal}(\mathrm{GF}(p^3))$. Hence $|U_1 : [U_1, S]| = p$. Since S/U_i is elementary abelian this implies (1). (2) is now obvious as $|S/\Phi(S)| = p^4$.

From (2.2), (2.3), and (2.4) we obtain the following reduction which will be useful for us:

(2.5) LEMMA. *Suppose* $\mathscr{P}$ *is a parabolic system of rank* 3 *with hyperbolic linear diagram* Δ *of the group* G, *satisfying* $B_G = 1$. *Then one of the following holds*:

(1) char $\mathscr{P} = 3$ *and* $m_2(H) \leqslant 2$.

(2) char $\mathscr{P} = 2$ *and* $m_r(H) \leqslant 2$ *for each prime* $r \mid |H|$ *and* $\Delta \neq \overset{0}{\circ}\!=\!\overset{1}{\circ}\!\equiv\!\overset{2}{\circ}$ *with* $\bar{G}_0 \simeq {}^2F_4(8)$.

Proof. Assume (2.5) is false. We first treat the case when some $\bar{P}_i$ is defined over GF(2). Then $H_i = H \cap \tilde{\bar{P}}_i \neq 1$, since $H = H_0H_1H_2$, where $I = \{0, 1, 2\}$. Hence $\bar{P}_i \simeq \mathrm{SU}_3(2)$ and Δ: $\underset{0}{\circ}\!=\!\underset{1}{\circ}\!\sim\!\sim\!\underset{2}{\circ}$ with $i = 0$ or 1. If P_2 is also solvable, then $\bar{P}_2 \simeq \mathrm{SU}_3(2)$, $\bar{G}_0 \simeq \bar{G}_2 \simeq U_5(2)$, and $i = 0$. Hence by (2.3) $\bar{G}_1 \simeq \mathrm{SU}_3(2) * \mathrm{SU}_3(2)$ and $H_0 = H_2$. But then (2) holds, a contradiction.

Hence $\bar{P}_2$ is non-solvable. If $i = 1$, then $\bar{G}_0 \simeq \bar{G}_2 \simeq U_5(2)$ and $\bar{G}_1 \simeq L_2(4) \times L_2(4)$. Further, by the structure of $U_5(2)$ we have $P_1/Q_1 \simeq GU_3(2) \times \mathbb{Z}_3$. Thus if we set $N_i = N_{P_i}(H)$ and $N = \langle N_i \mid i = 0, 1, 2 \rangle$, then condition $(*)$ of Theorem A of [10] is satisfied and so (B, N) is a BN pair of type Δ of G, a contradiction to [17, (2.7)] as in (2.2).

Thus $i = 0$, $\bar{G}_2 \simeq U_5(2)$, and $\bar{P}_1 \simeq L_2(4)$. Hence $\bar{G}_0$ is not defined over GF(2) and also is not isomorphic to $(P)\mathrm{SL}_3(4)$ since Δ is non-spherical. As $\bar{G}_1 \simeq \bar{P}_0 \times \bar{P}_2$ with $\bar{P}_2$ a simple rank 1 Lie-type group, condition $(*)$ is again satisfied in each group $\bar{G}_i$. Since $P_0/Q_0 \simeq \mathrm{GU}_3(2) \times C_H(\bar{P}_0)$, we obtain a BN pair as above, a contradiction.

Assume finally $\Delta = \overset{0}{\circ}\!=\!\overset{1}{\circ}\!\equiv\!\overset{2}{\circ}$ with $\bar{G}_0 \simeq {}^2F_4(8)$. Then by (2.4) condition $(*)$ of [10] is satisfied in each $\bar{G}_i$. Hence if $N_i = N_{P_i}(H)$ for $i = 1, 2$ and $N_0 \leqslant P_0$ such that $|N_0 : N_0 \cap B| = 2$ and (B, N^2) is a BN pair for G_2, where $N^2 = \langle N_0, N_1 \rangle$, then (B, N) is a BN pair of type Δ of G, a contradiction as above.

This shows that no $\bar{P}_i$ is defined over GF(2) if char $\mathscr{P} = 2$. Hence by (2.2) Δ: $\overset{0}{\circ}\!-\!\overset{1}{\circ}\!\equiv\!\overset{2}{\circ}$ with $\bar{G}_2 \simeq (P)\mathrm{SL}_3(4)$. But then $m_3(H) = 3$, since we assume (2.5) is false, and so $(P)\mathrm{GL}_3(4) \trianglelefteq G_2/M_2$ by (2.3), a contradiction to (2.2).

Thus char $\mathscr{P} = 3$ and, by (2.2), some $\bar{P}_i$ is defined over GF(3). Suppose Δ: $\overset{0}{\circ}\!\sim\!\sim\!\overset{1}{\circ}\!\sim\!\sim\!\overset{2}{\circ}$. Then, as $m_2(H) = 3$ but $m_2(C_H(\bar{G}_1)) = 1$ by (2.3), condition $(*)$ of Theorem A of [10] is satisfied in G_1/M_1. Hence by (2.4) $\bar{G}_i \simeq G_2(3)$ for $i = 0$ or 2 since otherwise condition $(*)$ of Theorem A of [10] is satisfied in each G_i/M_i, and so we obtain by [10] a BN pair of type Δ.

Assume without loss of generality $\bar{G}_0 \simeq G_2(3)$. Then $\bar{P}_1 \simeq \mathrm{SL}_2(3) \simeq \bar{P}_2$. If $\bar{P}_0$ is also defined over GF(3), then $\bar{G}_1 \simeq \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$ or $\mathrm{SL}_2(3) \times A_4$ by

(2.3). In any case $H_0H_2 \simeq \mathbb{Z}_2$ and (2.4) holds, a contradiction. Thus $\Delta = \overset{3^2}{\underset{0}{\circ}}\!\!-\!\!\!-\!\!\overset{3}{\underset{1}{\circ}}\!\!=\!\!\!=\!\!\overset{3}{\underset{2}{\circ}}$ or $\overset{3^3}{\underset{0}{\circ}}\!\!=\!\!\!=\!\!\overset{3}{\underset{1}{\circ}}\!\!=\!\!\!=\!\!\overset{3}{\underset{2}{\circ}}$ (with obvious notation!). Now a Cartan subgroup of $\bar{G}_2$ is non-cyclic, as otherwise $H_1 \leqslant H_0$. Hence $\bar{G}_2$ is in the first case a covering group of $\mathrm{PSU}_4(3)$ with non-cyclic Cartan subgroup. Since a Cartan subgroup of $\mathrm{SU}_4(3)$ is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_2$ and $Z(\mathrm{SU}_4(3)) \simeq \mathbb{Z}_4$, this implies $\bar{P}_0 \simeq \mathrm{SL}_2(9)$ and so by (2.3) $\bar{G}_1 \simeq \mathrm{SL}_2(9) * \mathrm{SL}_2(3)$. But the $H_2 \leqslant H_0$, a contradiction. In the second case $\bar{G}_1 \simeq \mathrm{SL}_2(3) \supsetneqq \mathrm{SL}_2(3^3)$ by (2.3). Hence in any case a Sylow 2-subgroup of H_0H_2 is isomorphic to $\mathbb{Z}_2$ and thus (2.5) holds. This final contradiction proves (2.5).

(2.6) LEMMA. *Suppose X is a universal rank* 1 *Lie-type group in* char p, *B a Borel subgroup of X, and $V = \langle C_V(B)^X \rangle$ a non-trivial* $\mathrm{GF}(p)X$*-module. Then the following hold*:

(1) *V is a direct sum of trivial and Steinberg* $\mathrm{GF}(p)X$*-modules, containing at least one non-trivial direct summand.*

(2) *$[V, x, x] \neq 0$ for each p-element x of X if p is odd.*

(3) *$[V, A, A] \neq 0$ for each four-subgroup A of X if $p = 2$.*

(4) *If V contains at least two Steinberg modules as direct summands, then $[H, x, x] \neq 0$ if p is odd resp. $[H, A, A] \neq 0$ if $p = 2$ for each hyperplane H of V.*

Proof. Let K be the algebraic closure of $\mathrm{GF}(p)$ and W a non-trivial irreducible KX-module satisfying $W = \langle C_W(B)^X \rangle$. Then by [5, (9.9)] W is the Steinberg module, since there is a unique such module. (This module has degree $|S|$, where $S \in \mathrm{Syl}_p(X)$, and is projective!) This shows that each non-trivial irreducible $\mathrm{GF}(p)X$-module V_1 satisfying $V_1 = \langle C_{V_1}(B)^X \rangle$ is a direct summand of W, if one considers W as a $\mathrm{GF}(p)X$-module.

Now let $Z = C_V(B)$ and V_0 be a $\mathrm{GF}(p)X$-submodule of V such tat $\bar{V} = V/V_0$ is irreducible. If $\bar{V}$ is trivial, then $\bar{V} = V_0 + C_V(X)$ by Gaschütz's theorem, since $V = V_0 + Z$. If $\bar{V}$ is non-trivial, then $\bar{V}$ is the $\mathrm{GF}(p)X$ Steinberg module as shown, whence $V = V_0 \oplus V_1$, $V_1 \simeq \bar{V}$ as $\mathrm{GF}(p)X$-module, since $\bar{V}$ is projective. So in any case

$$V = V_0 \oplus V_1, \qquad V_1 = \langle C_{V_1}(B)^X \rangle \text{ irreducible.}$$

Hence if $V_0 \neq 0$, then $Z \nleqslant V_1$ and thus $V_0 = \langle (Z \cap V_0)^X \rangle$, since $V_0 \simeq V/V_1$ as $\mathrm{GF}(p)X$-module. Proceeding by induction on dim V this proves (1).

(2) and (3) are now well known. ((3) follows already from the fact $\dim_K W = |S|$!) (4) is obvious, since either H contains some Steinberg submodule V_1 of V or $H + V_1 = V$ and $[V/V_1, x, x] \neq 0$ resp. $[V/V_1, A, A] \neq 0$, since V/V_1 again contains a Steinberg submodule.

A *failure-of-factorization module* (short FF module) in char p for the quasi-simple group X with $O_p(X)=1$ is a non-trivial $\mathrm{GF}(p)X$-module V satisfying

$$|A|\ |C_V(A)| \geqslant |V|$$

for some elementary abelian p-subgroup A of X. Such an A is called an offending subgroup. Directly from [2] we obtain:

(2.7) PROPOSITION. *Suppose G is a universal rank 2 Lie-type group in char p or $G \simeq \hat{A}_6$, $\hat{\Sigma}_6$ and $p=2$, $S \in \mathrm{Syl}_p(G)$, $B=N_G(S)$, and $P_i=U_iL_i$, $i=1,2$ the two maximal parabolic subgroups of G with unipotent radicals U_i and Levi complements L_i. Let $V=\langle C_V(S)^G\rangle$ be an FF module for G, satisfying*

(a) $C_V(G)=0$.

(b) *Suppose $P_1 \nleqslant N_G(Z)$, where $Z=C_V(S)$. Then $A \leqslant U_1$ for some quadratically acting offending subgroup.*

Then one of the following holds:

(1) *$G \simeq \mathrm{SL}_3(q)$ and V is the natural module.*

(2) *$G \simeq \mathrm{SL}_3(q)$, V is the direct sum of two natural modules, and $A=U_1$.*

(3) *$G \simeq \mathrm{Sp}(4,q)$ or A_6, V is the natural module.*

(4) *$G \simeq U_4(q)$ or $U_5(q)$, V is the natural module, and $A=Z(U_1)$.*

(5) *$G \simeq \Omega^-(6,q)$, $q=2^n$, V is the natural module, and $|A|=q^3$, $A \lhd O^{2'}(P_1)$.*

(6) *$G \simeq G_2(q)$, $q=2^n$, $|A|=q^3$.*

(7) *$G \simeq \hat{A}_6$, $\hat{\Sigma}_6$, $|V|=2^6$ and is the module obtained from $\hat{A}_6 \subseteq \mathrm{SL}_3(4)$. Further $A \lhd P_1$, $|A|=4$.*

Further in all cases, except when in cases (1) and (3) $|A|=q=|[V,A]|$, we have $V_1=\langle Z^{P_1}\rangle \leqslant [V,A]$.

Proof. This is a direct consequence of [2].

Without proof we list some properties of these modules, which are well known.

(2.8) COROLLARY. *Let G, V, and A be as in (2.7). Then the following hold*

(1) $[V, C_{U_1}([V,A])]=[V,A]$, except when $|A|=q=|[V,A]|$ *in cases (1) and (3) or $G \simeq U_5(q)$ in case (4).*

(2) $[V, A, U_1] \leqslant V_1$.

(3) $[V, A] = C_V(A)$, except *when* $|[V, A]| = q = |A|$ *in cases* (1) *and* (3) *or* $G \simeq U_5(q)$ *in case* (4).

(4) *Suppose* $P_1 \leqslant N_G([V, A])$. *Then case* (5) *or* (6) *of* (2.7) *does not hold. Further* $|A| > q$ *in cases* (1) *and* (3).

(5) *Suppose* $p = 2$. *Then* $[V, A] = [H, A]$ *for each* GF(2) *hyperplane* H *of* V, *except in the following cases*: (2.7)(1) *with* $q \leqslant 4$, (2) *with* $q = 2$, (3) *with* $q = 2$, *and* (7).

(2.9) LEMMA. *Let* G *be a universal rank* 2 *Lie-type group in* char p, $P = U \cdot L$ *a maximal parabolic subgroup of* G *with unipotent radical* U *and Levi complement* L *and* U^- *the opposite unipotent radical.* (*I.e.*, $L \leqslant N(U^-)$, $U \cap U^- = 1$!) *Suppose* $O^{p'}(L) = L_0 \simeq \mathrm{SL}_2(q)$, $q = p^r$, *where* $\mathrm{GF}(q)$ *is the field of definition of* G *as a Lie-type group, and* V *is an irreducible* $\mathrm{GF}(p)G$-*module such that* $C_V(U^-)$ *contains a natural* L-*submodule* V_0. *Then the following hold*:

(1) $V_0 = C_V(U^-)$

(2) $V = \langle V_0^U \rangle$.

Proof. (2) is a specialization of the corollary of [18], if we have shown (1). To prove (1) let $\bar{V}$ be the irreducible $\mathrm{GF}(q)G$-module, which, considered as $\mathrm{GF}(p)G$-module, contains V as a direct summand. Then by [5, (9.5)] $|C_{\bar{V}}(S)| = q$ for $S \in \mathrm{Syl}_p(G)$, since $\bar{V}$ is absolutely irreducible. But as $U^- L_0$ already contains a p-Sylow subgroup of G, we may assume $S \leqslant U^- L_0$ and thus

$$C_{\bar{V}}(S) = C_V(S) = C_{V_0}(S).$$

Hence $V = \bar{V}$ and [18] implies (1).

(2.10) LEMMA. *Let* G *be a universal rank* 2 *Lie-type group in* char 2 *or* 3 *different from* $A_2(q)$, $B = S \cdot H$ *a Borel subgroup, and* $W = N/H$ *a Weyl group of* G. *Let* w_0 *be the longest element of* W. *Then* $\langle w_0 \rangle = Z(W)$ *and the following hold*:

(1) *If* $P = U \cdot L$ *is a parabolic subgroup of* G *containing* B, *then* $P^{w_0} = U^- L$, *where* U^- *is the opposite unipotent radical. Further* $\langle U, w_0 \rangle = \langle U, U^- \rangle = G$. *Moreover, if* V *is an irreducible* $\mathrm{GF}(p)G$-*module, then* $C_V(U)$ *and* $V/[V, U]$ *are equivalent* L-*modules.*

(2) *If* char $G = 3$ *and* $F \leqslant H$ *is a four-subgroup, then* $w_0 \in C(F)$.

Suppose char $G = 2$ *for the next statements. Then*

(3) $H \simeq \mathbb{Z}_{q-1}^2$ *and* w_0 *inverts* H *in case* $G \simeq C_2(q)$, $G_2(q)$, *or* ${}^2F_4(q)$.

(4) *$H \simeq \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^3-1}$ and w_0 inverts H if $G \simeq {}^3D_4(q)$.*

(5) *$H \simeq \mathbb{Z}_{q+1} \times \mathbb{Z}^2_{q-1}$ and w_0 inverts $\mathbb{Z}^2_{q-1}$ and centralizes $\mathbb{Z}_{q+1}$ in case $G \simeq \mathrm{PSU}_4(q) \simeq \mathrm{SU}_4(q)$.*

(6) *$H \simeq \mathbb{Z}^2_{q+1} \times \mathbb{Z}^2_{q-1}$ and w_0 inverts $\mathbb{Z}^2_{q-1}$ and centralizes $\mathbb{Z}^2_{q+1}$ in case $G \simeq \mathrm{SU}_5(q)$. Further $|Z(G)| = (q+1, 5)$ in this case.*

Proof. Since $G \not\simeq A_2(q)$ we have $Z(W) \neq 1$. Hence $\langle w_0 \rangle = Z(W)$ and w_0 is $-id$ on the root system of G. This implies $U^{w_0} = U^-$ and $L^{w_0} = L$ by definition of U^-. Now $L\langle U, U^- \rangle$ is a parabolic properly containing P. Hence $L\langle U, U^- \rangle = G$ and $\langle U, U^- \rangle \trianglelefteq G$, which proves the first part of (1). The second follows immediately from the first and [18].
(2)–(4) are immediate consequences of [1, (7.22)], while (5) and (6) can easily be computed directly.

(2.11) PROPOSITION. *Suppose $\hat{G}$ is an extension of a universal rank 2 Lie-type group G in* char p, *$p = 2$ or 3, by a p'-group of inner and diagonal automorphisms. Let $S \in \mathrm{Syl}_p(G)$, $B = N_{\hat{G}}(S)$, and H be a complement to S in B. Suppose that the following hold:*

(*) *H is abelian. Further $m_2(H) \leqslant 2$ if $p = 3$ and $m_r(H) \leqslant 2$ for each prime $r \mid |H|$ if $p = 2$.*

Let $P = U \cdot L$ be a maximal parabolic subgroup G containing $B \cap G$ with unipotent radical U. Then H normalizes P and for each irreducible $\mathrm{GF}(p)\hat{G}$-*module V one of the following holds:*

(1) *If $p = 2$ and G is not of type $A_2(q)$ or ${}^3D_4(q)$, then $[V, h] \leqslant [V, U]$ for each $h \in C_H(C_V(U))$.*

(2) *If $G \simeq {}^3D_4(q)$, $q = 2^n$, then $[V, h] \leqslant [V, U]$ for each $h \in C_H(C_V(U)) \cap G$.*

(3) *If $p = 3$ and $G \not\simeq A_2(q)$, then $[V, t] \leqslant [V, U]$ for each involution $t \in C_H(C_V(U))$.*

Proof. Since all the statements in (2.11) are about elements of $C_H(C_V(U))$ we may assume $H = (H \cap G) C_H(C_V(U))$. But then V is already an irreducible $\mathrm{GF}(p)G$-module. Since H induces only inner or diagonal automorphisms on G, there exists a Weyl group $W = N/H \cap G$ of G normalizing H. Suppose G is not of type $A_2(q)$ and let $\langle w_0 \rangle = Z(W)$. Let $h \in C_H(C_V(U))$ be as in (1)–(3). Claim

(†) $\langle h \rangle^{w_0} = \langle h \rangle$.

In cases (2) and (3) this is obvious by (*) and (2.10). In case (1) we have $H = C_H(w_0) \times [H, w_0]$ with $[H, w_0] \leqslant H \cap G$ and H is thus inverted by w_0.

Since $m_r([H, w_0])=2$ for each prime $r \mid |H, w_0]|$ by (2.10), it follows that $(|C_H(w_0)|, |[H, w_0]|)=1$ by $(*)$. Hence (†) also holds in this case.

Now by the corollary of [18] we have $V=[V, U] \oplus C_V(U^-)$ and by (2.10) $U^- = U^{w_0}$. Hence by (†) $h \in C_H(C_V(U^-))$ and (2.11) holds.

(2.12) COROLLARY. *Let $\hat{G}$ be as in* (2.11) *with $G \simeq \mathrm{SL}_3(3)$ and let S, B, H, P, and U be as in* (2.11). *Suppose also that*

$(*)$ *$m_2(H) \leqslant 2$ and H is abelian*

holds. Then there exists no non-trivial irreducible GF(3)$\hat{G}$*-module satisfying*

(1) *$C_V(U)$ is a non-trivial module for P'.*

(2) *Some 3-element of G acts quadratically on V.*

(3) *There exists an involution $t \in C_H(C_V(U))$.*

Proof. By $(*)$ $t \in G$ and we may, to prove (2.12), assume $G=\hat{G}$. Since each 3-element of G fuses to an element of $P-U$, $C_V(U)$ contains a composition factor on which $L \simeq \mathrm{GL}_2(3)$ acts faithfully, a contradiction to (3). ($\mathrm{PGL}_2(3)$ admits no non-trivial GF(3)-module on which 3-elements act quadratically!)

(2.13) LEMMA. *Suppose $G \in \{G_2(q), G_2(2)', {}^3D_4(q), {}^2F_4(2), {}^2F_4(2)'\}$, $S \in \mathrm{Syl}_2(G)$, and P is a maximal parabolic containing S with $U=O_2(P)$. Then $A \leqslant U$ for each elementary abelian normal subgroup A of $B=N_G(S)$.*

Proof. We use the description of P in [17, (3.2)]. Suppose (2.13) is false. Then P is not the normalizer of a long root subgroup in case $G_2(q)$, $G_2(2)'$ or ${}^3D_4(q)$. Hence $P_0=U\langle A, A^x\rangle = O^{p'}(P)$ (resp. $P_0=U\langle A, A^x\rangle$ of index 2 in P in case ${}^2F_4(2)$). Let $U_0=[U, A][U, A^x]$, $Z=[U, A] \cap [U, A^x]$. Then $U_0 \lhd P_0$, $Z \leqslant Z(U_0)$, U_0Z is elementary abelian, and $[U, P_0] \leqslant U_0$. Hence again by [17, (3.2)] $U=U_0$ resp. $|U: U_0| \leqslant 2$ in case $G_2(2)$ or ${}^2F_4(2)$. But this is obviously impossible, since by [17, (3.2)] $Z(U)$ is a non-trivial P_0-module.

(2.14) LEMMA. *The following hold*:

(1) $m_2({}^3D_4(2))=5$, $m_2({}^2F_4(2)) \leqslant 6$.

(2) *Suppose V is a non-trivial* GF(2) ${}^2F_4(2)$ *or* ${}^2F_4(2)'$*-module. Then $|V: C_V(t)| \geqslant 2^8$ for each involution t in ${}^2F_4(2)$.*

(3) *Suppose V is a non-trivial* GF(2) ${}^3D_4(2)$*-module. Then $|V: C_V(t)| \geqslant 2^6$ and equality can only hold in case t is a long root involution.*

Proof. (1) is easy to see with [17, (3.2)]. (2) and (3) are contained in more general unpublished results of Cooperstein and Mason. But for the convenience of the reader we sketch a proof using the Weyl-submodule method of [4].

Let $G = {}^2F_4(2)$, $W_1 = W(G) \simeq D_{16}$. Then G has two classes of involutions, both fusing to W_1. Let $W = W(F_4(2)) \simeq (D_8 * D_8)(\Sigma_8 \times \Sigma_3)$ and embed $W_1 \subseteq W$. Let $W = \langle a, b, c, d \rangle$ with $\overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\!=\!=\!\overset{c}{\circ}\!-\!\!-\!\overset{d}{\circ}$. Then, with the embedding $G \subseteq F_4(2)$, $W^1 \subset W$ it is easy to see that one class of involutions of G fuses to $Z(W)^{\#}$ and the other to $O_2(W)a \cdot d$.

Now each irreducible GF(2)G-module is the restriction of an irreducible GF(2)$F_4(2)$-module $M(\lambda_J)$ to G, where λ_J is some basic weight. (I.e., $J \subseteq \{1, 2, 3, 4\}$.)

Let $N(\lambda_J) = \mathrm{GF}(2)\{\lambda_J^W\}$. Then $N(\lambda_J)$ is a W-submodule of $M(\lambda_J)$ and for each involution $t \in W$ we have by [4] the formula

$$\dim C_{N(\lambda_J)}(t) = \frac{|\{\lambda_J^W\}| + |\mathrm{Fix}(\{\lambda_J^W\}, t)|}{2},$$

where $\mathrm{Fix}(\{\lambda_J^W\}, t)$ is the number of fix points of t on $\{\lambda_J^W\}$. Now $\dim N(\lambda_J) = |\{\lambda_J^W\}| = |W : C_W(\lambda_J)| \geqslant 24 = 2^7 \cdot 3^2/2^4 \cdot 3$ since $C_W(\lambda_J)$ is a parabolic subgroup of W, while $|\mathrm{Fix}(\{\lambda_J^W\}, t)| \leqslant 8$ for each involution $t \in W_1$. Hence codim $C_{N(\lambda_J)}(t) \geqslant 8$, which proves (2). (These inequalities are rather crude. But we do not need better ones!)

Now (3) may be proved along the same lines, if one notices that ${}^3D_4(2)$ has two clases of involutions, the long and the short root elements, both fusing to $W^1 = W({}^3D_4(2)) \simeq D_{12}$, where the short root elements fuse to $Z(W^1)^{\#}$. Further each irreducible GF(8) ${}^3D_4(2)$-module is the restriction of an irreducible GF(8)$\Omega^+(8.8)$-module to ${}^3D_4(2)$.

(2.15) Lemma. *Let $G = G_2(2)$. Then the follwing hold:*

(1) *G has exactly three non-trivial irreducible* GF(2)*-modules. They are of dimension* 64, 14, *and* 6. (*The last one is called in this paper the natural module.*)

(2) *Let V be a non-trivial irreducible* GF(2)*G-module. Then $|V : C_V(t)| \geqslant 2^5$ for each involution t of G or V is the natural module.*

(3) *If V is the natural* GF(2)*G-module, then $|V : C_V(t)| = 4$ if t is a long root involution while $|V : C_V(t)| = 8$ if t is a short root involution.*

(4) *The irreducible* GF(2)*G-modules of* dim 6 *and* 14 *remain irreducible if they are restricted to $G' \simeq U_3(3)$. The* 64 dim *module splits into the direct sum of two irreducible ones.*

Proof. (1), (3), and (4) are well known. To prove (2) notice that $\dim V \geqslant 14$ and $C_V(S) \simeq \mathbb{Z}_2$ if V is not the natural GF(2)G-module and $S \in \mathrm{Syl}_2(G)$. Further S is generated by three short root involutions and a subgroup $P_0 \simeq (\mathbb{Z}_4 \times \mathbb{Z}_4)\Sigma_3$ with $C_V(P_0)=1$ is generated by three long root involutions.

(2.16) LEMMA. *Let $G=G_2(2)$ and V be the natural* GF(2)*G-module. Then the following hold*:

(1) *If $A<G$ is a quadratically acting elementary abelian subgroup of order* 8, *then $|C_V(A)|=|[V,A]|=8$ and A is uniquely determined up to conjugacy. Further $[V,A]=[V,C]$ for each hyperplane C of A.*

Choose A as in (1) *and in* (2)–(4). *Then*

(2) *Let $t \in A$ be a long root involution. Then $A \leqslant M_t = O_2(C(t))$ and $C_V(M_t) \leqslant [V,A]$.*

(3) *Let $B=A \cap G'$. Then*

$$|[H,B]|=2 \text{ or } 8 \qquad \text{for each hyperplane } H \text{ of } V.$$

(4) *Let W be an indecomposable* GF(2)*G-module with $W/C_W(G)=V$ and $C_W(G)\neq 0$. Then $|C_W(G)|=2$ and the following hold*:

(α) $|[W,A]|=16$ *and A acts quadratically on W.*

(β) $|[W,B]|=8$ *and $[W,A]=[W,C]$ for each hyperplane C of A different from B.*

(γ) $|[H,B]|=2$ *or* 8 *for each hyperplane H of W.*

Proof. Since all involutions of G' are long root involutions, $B^{\#}$ consists of all long root involutions of A. Thus if $t \in B^{\#}$, then $A \leqslant M_t$, since $C(t)/M_t$ acts as $L_2(2)$ on $[V,t]=C_V(M_t)$. This implies $A=M_t \cap M_\tau$ if t, τ are different involutions of B, which proves (1) and (2).

To prove (3) let $B=\langle t, \tau\rangle$. If $H=C_V(t)C_V(\tau)$, then $[H,B] \leqslant [V,t] \cap [V,\tau]$ and thus $|[H,B]|=2$. So we may assume $V=HC_V(t)$. Then, since $[C_V(t),\tau] \leqslant [V,t]$, we have $[H,\tau] \not\leqslant [V,t] = [H,t]$. But then $|[H,B]|=8$.

Now $C_W(G)\simeq \mathbb{Z}_2$ in (4) follows from [7]. Since G is generated by two conjugates of A we have $|[W,A]|=16$. As $W_0=[W,A]=C_W(G)[W,\sigma]$ for each $\sigma \in A-B$ this proves (α).

$|[W,t]|=4$ for each $t \in B^{\#}$ is obvious. Pick $\mathbb{Z}_3 \simeq H \leqslant N_G(B)$. Then $W_0=C_{W_0}(H)\times[W_0,H]$ and both factors have order 4. Since $C_{W_0}(H)\cap [W,t]\neq 0$ and $B^{\#}=t^H$ we obtain $|[W,B]|=8$. Now $A=BC_A[H]$ and $N_G(A)$ transitive on $A-B$ imply (β).

As $C_W(G) \not\leqslant [W,B]$ it is clear that (γ) is a consequence of (3) and (4)(β).

(2.17) LEMMA. *$\hat{\Sigma}_6$, the perfect extension of Σ_6 by $\mathbb{Z}_3$, has the following irreducible* GF(2)*-modules*:

(1) *Non-faithful: The trivial module, the two* 4*-dimensional* Sp(4, 2)*-modules, and the* 16*-dimensional Steinberg* Sp(4, 2)*-module.*

(2) *Faithful: Two* 6*-dimensional modules obtained from* $\hat{\Sigma}_6 \gtrsim \Gamma L_3(4)$, *namely the natural* $\Gamma L_3(4)$*-module and its dual. One* 18*-dimensional module, which is obtained in the following way: Let* V *be the natural* $\mathrm{GF}(4)\mathrm{SL}_3(4)$*-module,* V^* *its dual, and* σ *the non-trivial automorphism of* GF(4). *Then* $M = V \otimes_{F_4} V^*\sigma$ *is a* 9*-dimensional irreducible* $\mathrm{GF}(4)\mathrm{SL}_3(4)$*-module. If one considers* M *as* GF(2)*-module,* $\Gamma L_3(4)$ *acts on* M *and* M *remains irreducible restricted to* $\hat{\Sigma}_6$.'

(3) *All these modules, except the* 16*-dimensional one, which splits into the direct sum of two* 8*-dimensional irreducible modules, remain irreducible restricted to* $\hat{A}_6$. *Moreover, we obtain in this way all the irreducible* $\mathrm{GF}(2)\hat{A}_6$*-modules.*

Proof. (1) and (2) are obvious, since $\hat{\Sigma}_6$ has seven classes of elements of odd order. $\hat{A}_6$ has 10 classes of elements of odd order. Since the two 6-dimensional $\mathrm{GF}(2)\hat{A}_6$-modules correspond to four different 3-dimensional $\mathrm{GF}(4)\hat{A}_6$-modules, namely V, V^*, $V\sigma$, and $V^*\sigma$, (3) is also obvious.

(2.18) PROPOSITION. *Let* G *be a universal rank* 2 *Lie-type group in* char 2 *different from* ${}^2F_4(q)$ *or* $\hat{A}_6$, $\hat{\Sigma}_6$, V *a non-trivial irreducible* $\mathrm{GF}(2)G$*-module, and* $A \neq 1$ *an elementary abelian* 2*-subgroup of* G, *satisfying*

(i) $[V, A, A] = 0$

(ii) $2\,|A|\,|C_V(A)| \geq |V|$.

Then one of the following holds:

(1) $G/C_G(V) \in \{\mathrm{SL}_3(q), \mathrm{Sp}(4, q)$ (resp. A_6), $U_4(q)$, $U_5(q)$, $\Omega^-(6, q)$, $G_2(q)$ (resp. $G_2(2)')\}$ *and* V *is the natural* $\mathrm{GF}(2)G$*-module.*

(2) $G \simeq \hat{A}_6$ *or* $\hat{\Sigma}_6$ *and* $|V| = 2^6$.

Further, if $G \in \{U_4(q), U_5(q), \Omega^-(6, q), G_2(q)\}$ *and* $q > 2$, *then* A *is a subgroup of index at most* 2 *of an offending subgroup which is uniquely determined up to conjugacy.*

Before we start with the proof of (2.18) several remarks are in order.

(1) The assertion of (2.18) remains of course correct if we include $G \simeq {}^2F_4(q)$. But in this case we only need the result in the cases of ${}^2F_4(2)$ and ${}^2F_4(2)'$, in which it follows from (2.14).

(2) The $\mathrm{GF}(2)\mathrm{SL}_3(4)$-module obtained from $V \otimes_{F_4} V\sigma$, V as in (2.17) satisfies (ii), but not (i).

(3) Proposition (2.18) is just a small part of an (unpublished) thesis of Phil. McClurg at Santa Cruz. So we only sketch a proof.

Proof. Suppose neither (1) nor (2) of (2.18) holds. Then we may by (2.7) assume that V is not an FF-module (in the sense of (2.7)!). Further, by (2.17), $G \not\simeq \hat{A}_6, \hat{\Sigma}_6$. Choose A to be minimal satisfying (i) and (ii) and let $A \leqslant S \in \mathrm{Syl}_2(G)$, $B = N_G(S)$, $Z = C_V(S)$, and $P = U \cdot L$ be a maximal parabolic of G containing B not normalizing Z. Further, let $V_0 = C_V(U)$. We first show

$(\alpha)\langle A^G \cap P\rangle \leqslant U.$

Namely, suppose $B = A^g \leqslant P$, but $B \nleqslant U$. Then $C_V(B \cap U) \geqslant V_0 C_V(B)$ and $|V_0 : C_{V_0}(B)| \geqslant |B : B \cap U|$.

Hence the minimality of A implies $B \cap U = 1$. But then either $O^{2'}(L) \simeq \mathrm{SL}_2(q)$, $q = |B|$, $|V : V_0 C_V(B)| \leqslant 2$, or $V = V_0 C_V(B)$. Hence either $|A| = 2$ or V_0 is the only non-trivial $O^{2'}(P)$ composition factor of V. In the second case (2.10)(1) and [18] show $G \simeq \mathrm{SL}_3(q)$ and $|V| \leqslant 8q^3$, whence it is easy to see that (2.18) holds. In the first case it is well known (e.g. [20]) that (2.18) holds.

(α) shows that if $P_1 = U_1 L_1$ is the other maximal parabolic containing B, then $P_1 \leqslant N(Z)$. Next we show

(β) Suppose $A \neq B = A \cap A^g \neq 1$, $g \in G$, and let $C = \langle A, A^g\rangle$. *Then* $|C|\ |C_V(C)| \geqslant |V|$.

Namely the minimality of A implies $|A : B| > |C_V(B) : C_V(A)|$. Further obviously, $|C : A| \geqslant |A : B|$, $C_V(C) \geqslant C_V(A) \cap C_V(A^g)$, and $C_V(B) \geqslant C_V(A) \cdot C_V(A^g)$. Hence

$$|C : A| > |C_V(B) : C_V(A)| \geqslant |C_V(A) \cdot C_V(A^g) : C_V(A)| \geqslant |C_V(A) : C_V(C)|$$

and so $|C|\ |C_V(C)| \geqslant 2\ |A|\ |C_V(A)| \geqslant |V|$.

Especially, since we assume that V is no FF-module, (β) shows that either $[A, A^g] \neq 1$ or $A \cap A^g = 1$ for each $g \in G - N(A)$. Hence, if $G \simeq \mathrm{SL}_3(q)$, $\mathrm{Sp}(4, q)$, then (α) and the structure of P_1 and, if $G \simeq U_4(q)$, $U_5(q)$ and $O^{2'}(L) \simeq L_2(q^2)$, then [17, (3.2)] show that in any case either $|A| \leqslant q$ or $A = \Omega_1(U)$.

In the first case (2.18) is again easy to show, using generational properties of these groups. If in the second case $G \simeq \mathrm{SL}_3(q)$, then (2.18) is well known, while in the other cases, as $G = \langle U, U^- \rangle$ by (2.10), (2.18) is easy to see.

If finally $|C|\ |C_V(C)| \geqslant |V|$ for some subgroup C of U, then again Theorem A of [2] shows that (2.18) holds.

(2.19) Corollary. *Let G, V, and A be as in* (2.18). *Then one of the following holds*:

(1) $|[V, A]: [H, A]| \leqslant 2$ *for each* GF(2)*-hyperplane* H *of* V.

(2) $G \in \{L_3(q), \mathrm{Sp}(4, 2), A_6, \Omega^-(6, 2), G_2(2), G_2(2)', \hat{\Sigma}_6, \hat{A}_6\}$.

Proof. If $G \simeq U_4(q)$, $U_5(q)$ and V is the natural module or if $G \simeq G_2(q)$ and $q \geqslant 4$, (1) is obvious, since A contains an involution t with $[V, A] = [V, t]$. If $G \simeq \mathrm{SL}_3(q)$ there is nothing to show. Assume next $G \simeq \mathrm{Sp}(4, q)$. Then A is contained in a root group of transvections, if $q \geqslant 4$ and $|A| \leqslant q$, whence (1) holds. If $|A| > q$, then there exists again a $t \in A^{\#}$ with $[V, A] = [V, t]$.

So only the case $G \simeq \Omega^-(6, q)$, $q \geqslant 4$ remains to be treated. But then A contains by (2.18) a long root involution t and (2.19)(1) is obvious if $V = C_V(t) + H$. So assume $C_V(t) < H$. But then

$$[V, t] \leqslant [C_V(t), A] \leqslant [H, A]$$

by the action of $\Omega^-(6, q)$ on its natural module and (2.19)(1) holds again.

(2.20) LEMMA. *Let G be as in* (2.18), *A an elementary abelian 2-subgroup of G, and V a* GF(2)*G-module satisfying* (i) *and* (ii) *of* (2.18). *Suppose that in addition the following conditions are satisfied*:

(iii) $C_V(G) = 0$.

(iv) $V = \langle C_V(S)^G \rangle$, *where* $A \leqslant S \in \mathrm{Syl}_2(G)$.

(v) *V is not irreducible.*

Then one of the following holds:

(1) $G \simeq \mathrm{SL}_3(q)$ *and V is the direct sum of two equivalent natural* GF(2)*G-modules.*

(2) $G \simeq \mathrm{SL}_3(2)$ *and V is the direct sum of three equivalent natural modules.*

(3) $G \simeq \mathrm{Sp}(4, 2)$ *and V is the direct sum of two equivalent natural modules.*

(4) $G \simeq \mathrm{SL}_3(2)$, *$V$ is the extension of a natural* GF(2)*G-module by its dual. Further $C_V(P) \neq 0$ for each maximal parabolic P of G.*

Proof. Let V_0 be an irreducible GF(2)G-submodule of V and $\bar{V} = V/V_0$. Then by (iii) and (iv) V_0 and $\bar{V}$ are both non-trivial GF(2)G-modules. Further, both are FF-modules. Hence it is easy to see with (2.7) that $G \simeq \mathrm{SL}_3(q)$ or $\mathrm{Sp}(4, 2)$ and in the first case $\bar{V}$ contains only one non-trivial composition factor which is equivalent to V_0 if $q \geqslant 4$. Hence [15, (2.3)] implies that (1) holds in this case.

Next suppose $G \simeq \mathrm{Sp}(4, 2)$. Then by (iv) $\bar{V}/C_{\bar{V}}(G)$ is equivalent to V_0. Suppose the extension of $\bar{V}$ by $C_{\bar{V}}(G) = \bar{V}_1$ does not split. Now

$|A|=8$ or 2. In the first case $|V_1 \cap [V, A]|=8$, so that $C_{V_1}(N_G(A)) \neq 0$, a contradiction to $C_V(G)=0$ and Gaschütz's theorem. If $|A|=2$, then $[V, A, A^g]=0$ if $[A, A^g]=1$. Hence if $T=\langle A^G \cap S\rangle$, then $|C_V(T)|=2^5$ and $C_V(N_G(T)) \neq 0$, again a contradiction. Thus $\bar{V}=C_{\bar{V}}(G) \oplus [\bar{V}, G]$ and [13, Prop. 4] imply that V contains a submodule which is the direct sum of two equivalent natural modules. But then (iii) and (iv) show that (3) holds.

Next assume $G \simeq L_3(2)$ and all non-trivial G-composition factors are equivalent. Then, as $C_{\bar{V}}(S) \nleq C_{\bar{V}}(G)$, we have $\bar{V}=C_{\bar{V}}(G) \oplus [\bar{V}, G]$ and $[\bar{V} G]$ is by [15, (2.3)] the direct sum of at most two natural modules. But then again [15, (2.3)] shows that (2) holds.

In the final case V contains two non-equivalent natural $L_3(2)$-composition factors. Suppose $\bar{V}$ is not irreducible. If $\bar{V}$ is indecomposable then the structure of such a module shows $C_{\bar{V}}(S)=C_V(G)$, contradicting (iv). So $\bar{V}=C_{\bar{V}}(G) \oplus [\bar{V}, G]$. But then the co-image V_1 of $C_{\bar{V}}(G)$ is an indecomposable module of order 2^4 with $C_{V_1}(G)=0$, a contradiction to (ii). Thus $|V|=2^6$ and $C_V(P) \neq O$ remains to be shown.

To show this we may assume $C_{V_0}(P)=O$. Hence by (iv)

$$C_{\bar{V}}(P)=C_{\bar{V}}(S)=\overline{C_V(S)} \simeq \mathbb{Z}_2.$$

Let $U=O_2(P)$. Then $C_V(U)=C_{V_0}(U) C_V(S)$ and thus $C_V(U)=C_{V_0}(U) \times C_V(P)$, which proves (4).

(2.21) Lemma. *Let $R/Z \in \mathscr{L}_p^2$ with $1 \neq Z \leqslant R' \cap Z(R)$ a p-group, $S \in \mathrm{Syl}_p(R)$, and $S<P<R$ such that P/Z is a maximal parabolic of R/Z. Then the following hold*:

(1) *If $\Omega_1(Z(S)) \leqslant Z$, then* $R \simeq \mathrm{SL}_2(7)$, $\mathrm{SL}_2(9)$, or a covering group of $U_4(2)$

(2) *If $O^{p'}(P)$ is isomorphic to $O^{p'}(P_1)$, where P_1 is a maximal parabolic of some other element G of $\mathscr{L}_p^2$, then $R/Z \simeq \mathrm{PSL}_3(4)$ and $G \simeq \mathrm{Sp}(4,4)$.*

Proof. This is an easy consequence of the description of the covering groups of the elements of $\mathscr{L}_p^2$ in [3, p. 20 and 21].

(2.22) Lemma. *Let G be a group. Then G contains no pair of subgroups G_0, G_1 satisfying*:

(1) *$\bar{G}_i=G_i/Q_i \in \{U_4(q), U_5(q)\}$, $q=p^m$ where $Q_i=O_p(G_i)=F^*(G_i)$.*

(2) *Let $Z_i=\langle\Omega_1(Z(S_i))^{G_i}\rangle$, $S_i \in \mathrm{Syl}_p(G_i)$. Then $Z_i \nleq Z(G_i)$ for $i=0,1$ and if $p=2$, then $Z_0/C_{Z_0}(G_0)$ is not a natural $\Omega^-(6, q)$-module.*

(3) *$Z_i \leqslant G_j$ but $Z_i \nleq Q_j$ for $i \neq j$, $\{i, j\}=\{0,1\}$.*

Proof. Suppose (2.22) is false. Then we may by symmetry assume that

(∗) $|Z_1 : Z_1 \cap Q_0| \geqslant |Z_0 : Z_0 \cap Q_1|$.

Then $\tilde{Z}_0 = Z_0/C_{Z_0}(G_0)$ is an FF module for $\bar{G}_0$ and by [2] resp. (2.7) equality holds in (∗), so that also $\tilde{Z}_1$ is an FF module for $\bar{G}_1$. Hence by (2) and [2] $\tilde{Z}_i$ is a natural $\bar{G}_i$- module for $i = 0, 1$.

Now let $Z_0 \cap Q_1 < A_0 < Z_0$ such that $\bar{A}_0$ is a root group of transvections in $\bar{G}_1$. (A_0 exists!) Then

(†) $|\bar{A}_0| = q$, $|Z_1 : C_{Z_1}(A_0)| = q^2$, and $A_0 = C_{Z_0}(C_{Z_1}(A_0))$,

all by the action of $\bar{G}_1$ on its natural module. Hence, if $B_1 = C_{Z_1}(A_0)$, then $|\bar{B}_1| = q^2$ (in $\bar{G}_0$) and $|\tilde{Z}_0 : C_{\tilde{Z}_0}(\bar{B}_1)| = q^3$. But it is easy to see that this is impossible. ($\tilde{Z}_0$ is a GF(q^2) vector space!)

3. The Main Reduction. The Case $d \equiv 0$ (2)

We assume in this section that $\not{p}$ is a parabolic or quasiparabolic system of the group G of rank 3 and char p, with linear, non-spherical diagram $\Delta = \overset{0}{\circ}\sim\sim\sim\overset{1}{\circ}\sim\sim\sim\overset{2}{\circ}$. We use the notation and reduction of (2.1). Then by (2.2)

(1) $p = 2$ or 3. Further some $\bar{P}_i$ is defined over GF(2) or GF(3) or $\bar{G}_i \simeq (P)\mathrm{SL}_3(4)$, but $(P)\mathrm{GL}_3(4) \ntrianglelefteq G_i/M_i$ for $i = 0$ or 2.

We make the following further assumptions:

(2) $B_G = 1$

(3) $C_{\tilde{G}_i}(M_i) \leqslant M_i$ for $i = 0$, 1 and 2. (We are in the constrained case!)

The three-subgroup lemma implies $C_H(M_i) \leqslant C_H(\bar{G}_i)$ and thus $C_H(M_i) \leqslant C_H(\tilde{G}_i)$. Hence $C_H(M_i) \leqslant C_H(G) \leqslant B_G = 1$. Since $C_{G_i}(M_i) = C_{\tilde{G}_i}(M_i)C_H(M_i)$ by $G_i = \tilde{G}_i H$ this implies

(3′) $C_{G_i}(M_i) \leqslant M_i$ for $i = 0$, 1, and 2.

Finally we assume

(4) $G_2 \leqslant N(Z)$, where $Z = \Omega_1(Z(S))$ and $S \in \mathrm{Syl}_p(B)$.

We introduce the following notation:

$Z_i = \langle Z^{G_i} \rangle$, $i = 0, 1$. Then $Z_i \leqslant \Omega_1(Z(M_i))$.

$K_i = (P_2)_{G_i}$, $i = 0, 1$. Then $K_0 = B_{G_0}$

since $\bar{G}_0$ is quasisimple. Further $K_0/M_0 \leqslant Z(G_0/M_0)$. (As H is abelian!) By definition of parabolic systems

$$O^{p'p}(P_2/M_1) \leqslant K_1/M_1,$$

with equality holding if $\bar{P}_2$ is non-solvable. Hence

(*) $Z_1 = \langle Z^{K_1} \rangle = \langle Z^{P_2} \rangle < Z_0$ and $G_1 = (K_1 C_{G_1}(Z_1))B$ with $[K_1, C_{G_1}(Z_1)] \leqslant M_1$

since by the structure of $\bar{G}_1$ we have $[O^p(K_1/M_1), O^{p'p}(P_0/M_1)] = 1$, and since $O^{p'}(K_1/M_1) \cap Z(K_1/M_1) \leqslant (K_1/M_1)'$ and H is abelian.

Before introducing further notation on the coset graph of G_0, G_1 we show

(3.1). *Suppose $\bar{G}_0 \simeq (P)\mathrm{SL}_3(p^n)$. Then one of the following holds*:

(1) $\bar{G}_0 \simeq \mathrm{SL}_3(2)$

(2) $\bar{G}_0 \simeq (P)\mathrm{SL}_3(4)$ *or* $\mathrm{SL}_3(3)$, *but* Z_0 *contains no natural* $\bar{G}_0$*-submodule.*

Proof. It is obvious with Hypothesis (1) that $n \leqslant 3$. Further, if $n = 3$, then $\Delta = \circ\!\!-\!\!-\!\!\circ\!\!\equiv\!\!\circ$ and $\bar{G}_2 \simeq {}^3D_4(p)$, $p = 2$ or 3. Now (2.5) implies that condition (*) of [10] is satisfied in each $\bar{G}_i$, $i \leqslant 2$. Hence the obviously defined pairwise BN pair (in G_0, G_2!) is complete, a contradiction to [17, (2.7)] since B is finite.

Next suppose $\bar{G}_0 \simeq \mathrm{SL}_3(4)$ and Z_0 contains a natural $\bar{G}_0$-submodule Z'_0. Then $\Delta = \circ\!\!-\!\!-\!\!\circ\!\!\equiv\!\!\circ$, $\bar{P}_0 \simeq L_2(4)$ or $L_2(4^3)$, and by (*) there exists a subgroup $\mathbb{Z}_3 \simeq H_1 \leqslant H \cap C_{\bar{G}_1}(Z_1)$. But by (2.5) $H_1 \leqslant G_0$ and thus acts faithfully on $Z'_0 \cap Z_1$, a contradiction.

It is clear that $\bar{G}_0 \not\simeq \mathrm{SL}_3(9)$. Suppose finally $\bar{G}_0 \simeq \mathrm{SL}_3(3)$ and Z_0 contains a natural $\bar{G}_0$-submodule. Then the same argument as before implies $H \cap C_{\bar{G}_1}(Z_1) = 1$. But then by (*) $\bar{P}_0 \simeq \mathrm{PSL}_2(3)$. Hence the structure of the parabolics of the rank 2 Lie-type groups defined over GF(3) implies $\bar{G}_2 \simeq P\,\mathrm{Sp}(4, 3)$, a contradiction since Δ is non-spherical.

Let $\Gamma = \Gamma(G_0, G_1)$ be the coset graph of G_0, G_1 in G, and $d(\alpha, \beta)$, $\alpha, \beta \in \Gamma$ the usual distance metric on Γ. For $\alpha \in \Gamma$ let G_α, $\tilde{G}_\alpha$, $\bar{G}_\alpha$, Z_α, M_α, and K_α be the respective conjugates of the corresponding subgroup of G_i, $i = 0, 1$, and let

$$\Delta_i(\alpha) = \{\gamma \in \Gamma \mid d(\alpha, \gamma) = i\}, \qquad \Delta(\alpha) = \Delta_1(\alpha)$$
$$V_\alpha = Z_\alpha \Pi Z_\gamma, \qquad \gamma \in \Delta(\alpha).$$

Then it is obvious that K_α is the kernel of the action of G_α on $\Delta(\alpha)$.

Let $d = \min\{d(G_0, \delta) \mid Z_0 \nleqslant K_\delta\}$. Then, as S is finite, $d < \infty$, and since $Z_1 < Z_0$

$$d + 1 = \min\{d(G_1, \delta) \mid Z_1 \nleqslant K_\delta\}.$$

Note that by our assumptions $G = \langle \tilde{G}_i, \tilde{P}_i \rangle$ for $i \leqslant 2$. Hence

$$C_{Z_0}(\tilde{G}_0) = C_{Z_0}(\tilde{P}_2) = C_{Z_1}(\tilde{P}_2) = 1 \qquad \text{by (2).}$$

The main objective of this and the next section is to show that $d \leqslant 2$. For the rest of this section we assume

$$2 < d \equiv 0\ (2)$$

Then

$$R_\alpha = Z_\alpha \langle Z_\mu \mid \mu \in \Delta(\alpha) \cup \Delta_2(\alpha) \rangle \leqslant M_\alpha.$$

Let $\delta_0 = G_0$ and pick $\delta_d \in \Gamma$ with $d(\delta_0, \delta_d) = d$ and $Z_0 \nleqslant K_{\delta_d}$. A pair of vertices of Γ with such properties is called a *critical pair*. Choose a fixed arc

$$(\delta_0, \delta_1, ..., \delta_{d-1}, \delta_d).$$

Then by edge-transitivity of G on Γ we may assume $\delta_1 = G_1$. For vertices δ_i, $0 \leqslant i \leqslant d$ in this arc we simply write G_i, K_i, ... for the corresponding groups G_{δ_i}, K_{δ_i}, ... (Since the group $G_2 = \langle P_0, P_1 \rangle$ does not appear further on in this section this will not lead to confusion.) For vertices in $\Delta(\delta_0) - \delta_1$ resp. in $\Delta(\delta_d) - \delta_{d-1}$ we often write δ_{-1} resp. δ_{d+1} and G_{-1}, ... resp. G_{d+1}, ... for the corresponding groups.

We first show:

(3.2). *The following hold*:

(1) $Z_0 \leqslant M_i$, $Z_d \leqslant M_j$ *for* $0 \leqslant i \leqslant d-1$, $d \geqslant j \geqslant 1$, *and* $[Z_0, Z_d] \neq 1$.

(2) $\bar{G}_0 \simeq \bar{G}_d \in \{\mathrm{SL}_3(q), \mathrm{Sp}(4, q)\}$; Z_0 *is the natural* $\bar{G}_0$*-module and* $q = |Z_d : Z_d \cap M_0| = |Z_0 : Z_0 \cap M_d| = |[Z_0, Z_d]|$.

Proof. (1) Notice that $d \equiv 0$ (2) and the minimality of d imply $1 \neq [Z_0, Z_d] \leqslant Z_0 \cap Z_d$. As $M_i = O^{p'}(K_i)$ for i even it is obvious that (1) holds for i, j even. By $(*)$ and the structure of $\bar{G}_1$ we have $K_1 \cap M_1 M_0 = M_0$. Hence

$$Z_d \leqslant K_1 \cap M_1 M_2 = M_1,$$

which proves (1).

To prove (2) assume by symmetry that

$$|\bar{Z}_d| = |Z_d : Z_d \cap M_0| \geqslant |Z_0 : Z_0 \cap M_d| = |Z_0 : C_{Z_0}(\bar{Z}_d)|.$$

Then Z_0 is an FF module for $\bar{G}_0$ with offending subgroup $\bar{Z}_d$ and one of the cases of (2.7) holds for $\bar{G}_0, Z_0$. (2.22) implies that $\bar{G}_0$ is not $U_4(q)$ or $U_5(q)$ and Z_0 the natural $\bar{G}_0$-module. Now by (2.7) either (2) holds or

$$F = [Z_0, Z_d] \geqslant Z_1;$$

and we may assume the latter is the case. Claim

(†) $R_d \nleqslant M_1$.

Suppose (†) is false. Then, as R_d centralizes F and $\bar{Z}_d$, (2.8) implies $[Z_0, R_d] \leqslant F \leqslant Z_d$ or (2) holds. Hence $H_d = \langle Z_0^{G_d} \rangle$ centralizes R_d/Z_d. Since H_d is transitive on $\Delta(\delta_d)$ this implies $R_d = V_{d-1} = V_\lambda$ for each $\lambda \in \Delta(\delta_d)$. But then $V_{d-1} = V_{d-3} \leqslant M_0$, a contradiction.

As $R_d \leqslant C_{G_2}(Z_1)$ we have $R_d \leqslant G_1 \cap G_2$. By (†) and (∗) $G_1 = (K_1 L_1)B$, where $L_1 = \langle Q_2, R_d \rangle$. Hence L_1 is transitive on $\Delta(\delta_1)$ and normalizes F, since by (2.8)

$$[F, Q_2] \leqslant Z_1 \leqslant F.$$

Hence

$$F \leqslant F_1 = \bigcap Z_\mu,\ \mu \in \Delta(\delta_1).$$

Now as $F_1 \leqslant Z_2 \leqslant C(Z_d)$, (2.8)(3) implies in any case $F = F_1$. Hence $d > 4$ and so $R_d' = 1$, since otherwise $F \lhd G = \langle G_1, G_3 \rangle$. Moreover, as $F = F_1 \lhd P_2$, (2.8)(4) implies that one of the following holds:

(α) $F = Z_1$, $\bar{G}_0 \simeq \mathrm{SL}_3(q)$, $\mathrm{Sp}(4, q)$ or $\hat{A}_6$, and either Z_0 is the natural $\bar{G}_0$-module or $\bar{G}_0 \simeq \mathrm{SL}_3(q)$ and Z_0 is the direct sum of two natural modules.

(β) $\bar{G}_0 \simeq \hat{\Sigma}_6$, $|Z_0| = 2^6$, $|F| = 2^4$, and $|Z_1| = 4$.

Let $K = H \cap L_1$. Then K centralizes Z_1 and thus centralizes Z_0, since by (2.4) $\mathrm{GL}_3(4)$ or $\mathrm{GL}_3(3)$ is not involved in G_0/M_0. Hence (2.6) implies that in case $p = 3$, L_1/M_1 contains no quadratically acting three-element on V_1/Z_1, while in case $p = 2$, L_1/M_1 contains no quadratically acting four-group. As $[V_1, R_d, R_d] = 1$ by $R_d' = 1$ this shows that $p = 2$ and $|R_d\colon R_d \cap M_1| = 2$. Since $[R_d \cap M_1, Z_0] \leqslant F \leqslant Z_d$ and $|Z_0\colon Z_0 \cap M_d| \geqslant 4$ it now follows easily as in the proof of (†) that $\bar{G}_0 \simeq \bar{G}_d \simeq L_3(2)$ and Z_0 is the direct sum of two natural $\bar{G}_0$-modules. Especially $|Z_\lambda Z_d/Z_d| = 4$ for each $\lambda \in \Delta_2(\delta_d)$ and, as $|Z_\lambda Z_d\colon Z_\lambda Z_d \cap M_1| \leqslant 2$, there is a G_λ-invariant submodule Z_λ^1 of Z_λ contained in M_1. But then $Z_1 \leqslant C_{\bar{G}_\lambda}(Z_\lambda^1) = M_\lambda$ and thus $Z_\lambda \leqslant M_1$ for each $\lambda \in \Delta_2(\delta_d)$, a contradiction to (†). This proves (3.2).

(3.3). *Suppose $d > 4$. Then the following hold:*

(1) $R_d \cap G_1 \leqslant G_0$ *if* $p=3$.

(2) $|R_d \cap G_1 : R_d \cap G_0| \leqslant 2$ *if* $p=2$.

Proof. As $d>4$, $R_d \cap G_1$ acts quadratically on V_1. Let $H^1 = H \cap C_{G_1}(Z_1)$. Then by (3.1), (3.2) $[Z_0, H^1]=1$ and since by $(*)$ $V_1 = \langle Z_0^{C_{G_1}(Z_1)} \rangle$, (2.6) implies that $\tilde{V}_1 = V_1/Z_1$ is the direct sum of trivial and Steinberg modules for $X/C_X(\tilde{V}_1)$, where $X = C_{G_1}(Z_1)S$. Hence (2.6) implies (3.3).

(3.4). *Suppose* $\bar{G}_0 \simeq \mathrm{Sp}(4, q)$ *and set*

$$R_{0,1} = \langle Z_\lambda \mid \lambda \in \Delta_2(\delta_0), Z_\lambda \cap Z_1 = 1 \rangle.$$

Then the following hold:

(1) $R_{0,1} \leqslant C(Z_2)$

(2) $R_0 = F_0 R_{0,1}$, *where* $Z_0 \leqslant F_0 < R_0$, $F_0 \lhd G_0$, *and* R_0/F_0 *is a non-trivial irreducible* $\bar{G}_0$*-module.*

(3) $[R_{0,1}, Z_d] \nleqslant Z_0$.

Proof. (1) is obvious from the definition of $R_{0,1}$. Let $Z_0 \leqslant F_0 < R_0$ such that $F_0 \lhd G_0$ and R_0/F_0 is an irreducible $\bar{G}_0$-module. Then we may assume $Z_2 \nleqslant F_0$. As $O^p(\tilde{P}_2) \leqslant G_2 \cap G_0$ and acts irreducibly on Z_2/Z_1 this implies $Z_2 \cap F_0 = Z_1$ and $[R_0, O^p(\tilde{G}_0)] \nleqslant F_0$. Hence R_0/F_0 is a non-trivial $\bar{G}_0$-module and (2) holds by (2.9) since by definition $R_0 = \langle R_{0,1}^{G_0} \rangle$ and since $P_2 = Q_2(P_2 \cap P_{-2})$ for some $\delta_{-1} \in \Delta(\delta_0)$ with $Z_{-1} \cap Z_1 = 1$, and $P_{-2} = G_0 \cap G_{-1}$. (3) is now obvious.

(3.5). *Suppose* $\bar{G}_0 \simeq \mathrm{Sp}(4, q)$. *Then* $Z_1 = C_{Z_0}(R_{d,d-1})$ *and* $Z_{d-1} = C_{Z_d}(R_{0,1})$. ($R_{d,d-1}$ *defined similarly as* $R_{0,1}$).

Proof. Suppose (3.5) is false for Z_0. Then $R_{d,d-1} \nleqslant M_1$, since otherwise $C_{Z_0}(R_{d,d-1}) > Z_1$, and so $[Z_0, R_{d,d-1}] = [Z_0, Z_d]$, a contradiction to (3.4) (3). If now $R_{d,d-1} \leqslant C(Z_1)$, then

$$Z_1 < C_{Z_0}(R_{d,d-1}) \leqslant \bigcap_{\lambda \in \Delta(\delta_1)} Z_\lambda < Z_0,$$

since $\langle Q_0, R_{d,d-1} \rangle$ already acts transitively on $\Delta(\delta_1)$. But this is obviously impossible.

Hence $C_{Z_1}(R_{d,d-1}) = [Z_0, Z_d]$ and if $A = [Z_0, Z_d][Z_2, R_{d,d-1}]$ then $|A| \geqslant q^2$. Moreover, $A \leqslant Z_2 \cap Z_d \leqslant C_{Z_d}(R_{0,1})$. As above $|C_{Z_d}(R_{0,1})| \leqslant q^2$. Hence $A = C_{Z_d}(R_{0,1})$. Now

$$[Z_0, Z_d] \leqslant A < Z_d \cap M_0 = [Z_0, Z_d]^\perp,$$

considering Z_d as a symplectic space. As $A=[Z_0, Z_d][Z_2, Z_\mu]$ for some $\mu \in \Delta_2(\delta_d)$, A is a GF(q)-subspace. Hence $A=Z_\rho$ for some $\rho \in \Delta(\delta_d)$.

Suppose $\rho \neq \delta_{d-1}$. Then $Z_d \cap M_0 = AZ_{d-1}$ and $Z_d \cap M_0 \leqslant M_\mu$ for each $\mu \in \Delta(\delta_0)$. Hence

$$[R_{0,1}, Z_d \cap M_0] = [R_{0,1}, Z_{d-1}] \leqslant Z_0 \cap Z_{d-1} \leqslant C_{Z_0}(R_{d,d-1}).$$

Suppose $[R_{0,1}, Z_{d-1}] \leqslant [Z_0, Z_d]$. Then $R_{0,1} \leqslant G_{d-1}$ and, since $R_{0,1}M_{d-2}/M_{d-2}$ is generated by transvections on Z_{d-2}, $[R_{0,1}, Z_{d-1}] = [R_{0,1}, Z_d \cap M_0] = 1$. But then $[R_{0,1}, Z_d] = [Z_0, Z_d] \leqslant Z_0$, a contradiction to (3.4).

Thus, as above,

$$B = [Z_0, Z_d] \cdot [R_{0,1}, Z_{d-1}] = C_{Z_0}(R_{d,d-1}) = Z_\lambda \qquad \text{for some } \lambda \in \Delta(\delta_0).$$

By assumption $\lambda \neq \delta_{d-1}$. Pick $\sigma \in \Delta(\delta_0)$ with $Z_\sigma \cap Z_\lambda = 1$. Then

$$[V_\sigma, Z_d \cap M_0] \leqslant Z_\sigma \cap C_{Z_0}(R_{d,d-1}) = 1.$$

Hence, as before, $[V_\sigma, Z_d] \leqslant Z_0$ and thus $[R_{0,\lambda}, Z_d] \leqslant Z_0$, a contradiction to (3.4). This shows $A = Z_{d-1}$. But then

$$[Z_2, V_\lambda] \leqslant Z_2 \cap Z_\lambda \leqslant C_{Z_d}(R_{0,1}) \cap Z_\lambda = Z_{d-1} \cap Z_\lambda = 1$$

for each $\lambda \in \Delta(\delta_0)$ with $Z_{d-1} \cap Z_\lambda = 1$. Hence $[Z_2, R_{d,d-1}] = 1$, a contradiction. This proves (3.5).

(3.6). *Suppose* $d > 4$ *and* $\bar{G}_0 \simeq \mathrm{Sp}(4, q)$. *Then* $Z_1 = C_{Z_0}(R_d)$ *and* $C_{Z_d}(R_0) = Z_{d-1}$.

Proof. Suppose (3.6) is false for Z_0. Then by (3.5) $C_{Z_0}(R_d) = [Z_0, Z_d]$. By (3.3), (3.4) $|R_{d,d-1} : R_{d,d-1} \cap M_1| \leqslant 2$. Further $[Z_0, R_{d,d-1} \cap M_1] \leqslant [Z_d, Z_0]$ and $Z_0 \cap M_d \leqslant C(R_{d,d-1} \cap M_1)$. Hence $Z_0 \cap M_d$ centralizes a $GF(2)$-hyperplane of V_λ for each $\lambda \in (\delta_d)$ with $Z_{d-1} \cap Z_\lambda = 1$. But then $[Z_0 \cap M_d, V_\lambda] \leqslant V_1 \cap Z_\lambda \leqslant Z_{d-1} \cap Z_\lambda = 1$ by (3.4). This shows $R_{d,d-1} \leqslant C(Z_0 \cap M_d)$, whence $R_{d,d-1} \leqslant M_1$ as before, a contradiction to (3.5).

(3.7). $\bar{G}_0 \simeq \mathrm{SL}_3(q)$.

Proof. Suppose (3.7) is false. Then by (3.5) $R_{d,d-1} \leqslant G_1$, $R_{0,1} \leqslant G_{d-1}$, and, even in case $d=4$, $[R_{0,1}, V_1] = 1 = [R_{d,d-1}, V_{d-1}]$. Pick $\delta_{-1} \in \Delta(\delta_0)$ with $Z_{-1} \cap Z_1 = 1$. Then $[V_{-1}, V_{d-1} \cap M_{-1}] \leqslant C_{Z_{-1}}(R_{d,d-1}) = 1$. If $Z_d \cap M_0 \leqslant M_{-1}$ for each such δ_{-1} then arguing as in (3.5),

$$[R_{0,1}, Z_d] = [Z_0, Z_d],$$

a contradiction to (3.4). Thus we may assume $Z_d \cap M_0 \nleqslant M_{-1}$.

Now, as $V_{-1} \leqslant M_{d-2}$ by the above and so $[V_{-1}, V_{d-1} \cap M_0] \leqslant F_{d-2} \leqslant C(R_{d-2})$, $V_{d-1} \cap M_0$ acts quadratically on V_{-1} even in case $d=4$, and (2.6) implies $|V_{d-1} \cap M_0 : V_{d-1} \cap M_{-1}| = 2$. Hence $V_{d-1} \cap M_0 = (V_{d-1} \cap M_{-1})(Z_d \cap M_0)$. Since $\langle Q_{-2}, x \rangle$ is already transitive on $\Delta(\delta_{-1})$ for $\delta_{-2} \in \Delta(\delta_{-1}) - \delta_0$ and $x \in Z_d \cap M_0 - M_1$ we have $C_{Z_{-2}}(x) = Z_{-1}$.

Thus $|[V_{-1}, Z_d \cap M_0]| \geqslant q^2$. Hence $V_{-1} \nleqslant M_{d-1}$, since otherwise

$$Z_{d-1} = [V_{-1}, Z_d] = [V_{-1}, Z_d \cap M_0] \leqslant V_{-1},$$

a contradiction to $Z_d \nleqslant G_{-1}$. Arguing as in (3.5) this implies $Z_d \cap M_{-1} = Z_{d-1}$, whence $q=2$.

Suppose $|V_{d-1} : V_{d-1} \cap M_0| > 2$. Then $Z_1 = [Z_0, V_{d-1}] = Z_{d-1}$ and $[R_{0,1} \cap M_{d-1}, V_{d-1}] \leqslant Z_0$, a contradiction to (3.4)(2), since by (3.3) $|R_{0,1} : R_{0,1} \cap M_{d-1}| \leqslant 2$. This implies $|V_{d-1} : C_{V_{d-1}}(V_{-1})| = 4$, whence $\bar{P}_0 \simeq \Sigma_3$ and $\bar{G}_1 \simeq \Sigma_3 \times \Sigma_3$ since $M_0 \nleqslant M_{-1}$.

Thus setting $L_{-1} = C_{G_{-1}}(Z_{-1})$ it is easy to see that one of the following holds:

(1) V_{-1}/Z_{-1} is the direct sum of two natural $G_0 \cap G_{-1}/M_0 M_{-1}$-modules

(2) V_{-1}/Z_{-1} is the direct sum of three natural $G_0 \cap G_{-1}/M_0 M_{-1}$-modules, and $1 \neq C_{-1}/Z_{-1} = C_{V_{-1}/Z_{-1}}(L_{-1})$.

Suppose first (2) holds. Then $C_{-1} \lhd G_{-1}$ and $C_{-1} \leqslant C(Z_d \cap M_0)$. But then as before $[C_{-1}, Z_d] = [Z_0, Z_d] \leqslant Z_0$ and $G_0 = \langle G_0 \cap G_{-1}, Z_d \rangle \leqslant N(Z_0 C_{-1})$. But then $[G_0 \cap G_{-1}, C_{-1}] \leqslant Z_0 \cap C_{-1} \leqslant Z_{-1}$, a contradiction to (2).

So (1) holds. Especially, $V_1 = Z_0 Z_2$. Let $X = G_1 \cap G_0 \cap G_{-1}$. Then $X/M_0 \simeq \Sigma_3$ and $G_0 \cap G_1 = M_0 \langle Z_d^X \rangle X$. Suppose $d=4$. Then $[V_{-1}, Z_d] \leqslant [R_{0,1}, V_3] \leqslant [M_2 M_3, V_3] \leqslant Z_2 \leqslant V_1$ and thus $G_0 \cap G_1 \leqslant N(V_1 V_{-1})$. Since by symmetry also $G_0 \cap G_{-1} \leqslant N(V_1 V_{-1})$ we obtain $R_0 = V_1 V_{-1}$, a contradiction to $R_0' \neq 1$.

Thus $d > 4$ and by (3.3), (3.6) $|R_d : R_d \cap G_0| = 2$ and $[R_d \cap G_0, Z_0] \leqslant Z_1$. Hence (2.20) shows that $(R_0/Z_0)/C_{R_0/Z_0}(\bar{G}_0)$ is irreducible as a $\bar{G}_0$-module, since case (3) of (2.20) cannot hold. Hence by (2.18) R_0/Z_0 is the extension of a natural by a trivial $\bar{G}_0$-module. Especially, if $A/Z_0 = C_{R_0/Z_0}(Z_d)$, then $|R_0 : A| = 2$. Suppose $A \neq R_0 \cap M_{d-1}$. Then $Z_{d-2} = Z_{d-1}[A, Z_d]$, by (3.6) a contradiction to $Z_1 \cap Z_{d-1} \neq 1$. Hence either $Z_1 = [A, Z_d] = Z_{d-1}$ or $A \leqslant Z_0 M_d$. In the first case

$$[V_{-1}, Z_d \cap M_0] \leqslant Z_0 \cap V_{d-1} \leqslant Z_1 = Z_{d-1}$$

and thus $V_{-1} \leqslant M_{d-1}$, since $\langle V_{-1}, O_2(G_d \cap G_{d-1}) \rangle$ normalizes $Z_d \cap M_0$. But then $[V_{-1}, Z_d] \leqslant Z_{d-1} = Z_1$, a contradiction to $Z_d \nleqslant G_{-1}$.

In the second case $|R_0 : C_{R_0}(Z_d)| \leqslant 4$. On the other hand $C_{V_{-1}}(Z_d \cap M_0) \leqslant Z_0$ and so $|V_{-1} : C_{V_{-1}}(Z_d)| \geqslant 8$, a contradiction. This finally proves (3.7).

Summarizing the results of this section we obtain:

(3.8) THEOREM. *Suppose (with all the notations introduced in this section) that d is even. Then one of the following holds*:

(1) $d = 2$ *or*

(2) $\bar{G}_0 \simeq \mathrm{SL}_3(2)$ *and* Z_0 *is the natural* $\bar{G}_0$*-module.*

4. THE CASE d ODD

In this section we carry on with the hypothesis and notation of Section 3, but we assume $d \equiv 1$ (2) and, by way of contradiction, $d \geqslant 3$.

Let again (δ_0, δ_d) be a critical pair and $(\delta_0, \delta_1, ..., \delta_{d-1}, \delta_d)$ a fixed arc from δ_0 to δ_d. Then $Z_0 \nleqslant M_d$, but $Z_0 \leqslant M_{d-1}$ as $Z_0 \leqslant K_{d-1}$ and $M_{d-1} = O^{p'}(K_{d-1})$ since $\delta_{d-1} \sim \delta_0$. Set $L_d = M_d \langle Z_0^{G_d} \rangle$. Then we have

(4.1). *The following hold*:

(1) $O^{p'p}(G_d) \leqslant K_d L_d$, $[K_d, L_d] \leqslant M_d$.

(2) L_d *acts transitively on* $\Delta(\delta_d)$.

(3) $[V_1, V_d] \leqslant V_1 \cap V_d$ *but* Z_0 *acts non-trivially on* $V_d/[V_d, M_d] = \tilde{V}_d$.

(4) $M_0 M_1 = Q_2$ *or* $|Q_2 : M_0 M_1| = 2$ *and* $\bar{G}_0 \simeq {}^2F_4(2)$, $G_2(2)$, $G_2(2)'$, *or* Σ_6.

Proof. The first part of (1) is obvious by the structure of $\bar{G}_1$ and (*) of Section 3. The second part of (1) follows from $L_d \leqslant C_{G_d}(Z_d)$ and (*) of Section 3. (2) is an immediate consequence of (1). The first part of (3) is obvious by minimality of d. To prove the second part assume that $[\tilde{V}_d, Z_0] = 1$. Then, as $V_d = \langle Z_{d-1}^{L_d} \rangle$ by (2), $V_d = [V_d, M_d] Z_{d-1}$. This implies $V_d = [V_d, M_d; r] Z_{d-1}$ for arbitrary $r \in \mathbb{N}$, whence $V_d = Z_{d-1}$, a contradiction.

As $M_{d-1} \nleqslant M_d$ also $M_0 \nleqslant M_1$. Further by the structure of $\bar{G}_1$ we have $[O^{2'2}(\bar{P}_2), \bar{Q}_2] = 1$. Hence if $M_0 M_1 < Q_2$ then P_2/M_0 has a factor group $P_2/M_0 M_1$ with

$$1 \neq Q_2/M_0 M_1 \leqslant Z(O^{p'}(P_2/M_0 M_1)).$$

Hence the description of the structure of the parabolics of the rank 2 Lie-type groups in [17, (3.2)] shows that (4) holds.

Next we show:

(4.2) $p=2$

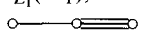
Proof. Suppose $p=3$ and change notation such that (δ_{d+1}, δ_1) is critical and pick $\delta_0 \in \Delta(\delta_1) - \delta_3$. Then all properties of (4.1) hold for $L_1 = M_1\langle Z_{d+1}^{G_1}\rangle$. Since Z_{d+1} acts quadratically on $\tilde{V}_1$ (2.6) applied to the action of some $(P)\mathrm{SL}_2(3) \gtrsim L_1/M_1$ implies

$$H^1 = H \cap L_1 \cap C(L_1/M_1) \simeq \mathbb{Z}_2.$$

Let $Z_0' \leqslant Z_0$ be an irreducible $\bar{G}_0$-module. Then Z_0' is non-trivial and $Z_0' \neq V_1' = \langle Z_0'^{G_1}\rangle$. Arguing as in (4.1) it follows that Z_{d+1} acts non-trivially on $\tilde{V}_1' = V_1'/[V_1', M_1]$. On the other hand, since by (2.4) $m_2(H) \leqslant 2$, (2.11), (2.12), and (3.1) imply that one of the following holds:

(α) $[Z_0', H^1] \leqslant [Z_0', M_1]$

(β) $\bar{G}_0 \simeq L_3(3)$ and Z_0' admits no quadratically acting three-element.

(As $H^1 \leqslant C(Z_1)$!) In case (α) $H^1 \leqslant C_{L_1}(\tilde{V}_1')$, since $\tilde{V}_1' = \langle(\tilde{Z}_0')^{L_1}\rangle$, a contradiction to (2.6). In case (β) $\Delta =$  and (2.5) and [10] show that $L_1/M_1 \simeq \mathrm{SL}_2(3)$, since otherwise $\bar{G}_2 \simeq {}^3D_4(3)$ and thus by (2.4) condition (*) of [10] is satisfied in each $\bar{G}_i$ and so the obvious pairwise BN pair is complete, a contradiction to [17, (2.7)]. Hence

$$|Z_0 : Z_0 \cap M_d| \leqslant |Z_{d+1} : Z_{d+1} \cap M_1| = 3.$$

But $Z_0 \cap M_d \leqslant M_{d+1}$. Since $L_1 = M_1\langle M_0, Z_{d+1}\rangle$ this shows that $|Z_0' : C_{Z_0'}(H^1)| \leqslant 3$, which obviously contradicts $H^1 \leqslant \tilde{G}_0$ and $\bar{G}_0 \simeq (P)\mathrm{SL}_3(3)$. (As $m_2(H) \leqslant 2$!)

(4.3). $|V_d : V_d \cap M_1| \leqslant 2 \geqslant |V_1 : V_1 \cap M_d|$.

Proof. Suppose (4.3) is false for V_d. Then $L_1 = M_1\langle V_d^{G_1}\rangle$ is not defined over GF(2) and $H^1 = H \cap L_1 \neq 1$. Now $[Z_1, H^1] = 1$. If $[Z_0', H^1] \leqslant [Z_0', M_1]$ for an irreducible $\bar{G}_0$-submodule Z_0' of Z_0, then arguing as in (4.2), (2.6) implies that V_d does not act quadratically on $\tilde{V}_1'$, a contradiction to $d > 2$. ($\tilde{V}_1'$ defined as in (4.2).)

So H^1 does not act trivially on $Z_0'/[Z_0', M_1]$. Hence (2.11) implies that $\bar{G}_0 \simeq {}^3D_4(q)$ or $(P)\mathrm{SL}_3(q)$. In the first case Hypothesis (1) implies $q \leqslant 4$. But if $q=4$ then easily $\bar{G}_2 \simeq (P)\mathrm{SL}_3(4)$ and $\mathbb{Z}_3 \simeq H^1 \leqslant \tilde{G}_0$, a contradiction to (2.11)(2). Hence $q=2$ and $H^1 \simeq \mathbb{Z}_7$ in the first case, since H^1 does not act trivially on Z_0'. But then either $\bar{G}_2$ is defined over GF(8) or $\bar{G}_2 \simeq {}^3D_4(2)$. In any case each $\bar{G}_i$ satisfies by (2.4) condition (*) of [10] and thus the obviously defined pairwise BN pair (defined in G_0, G_2!) is complete, a contradiction as in (3.7).

In the second case Hypothesis (1) implies $\tilde{G}_0 H^1/M_1 \simeq \mathrm{SL}_3(4)$ or $\mathrm{PSL}_3(4) \times \mathbb{Z}_3$, since in case $\bar{G}_0 \simeq L_3(2)$ obviously H^1 must centralize Z_0'. But then, as $[\tilde{P}_2, H^1] \leqslant Q_2$, $[\tilde{G}_0, H^1] \leqslant M_0$ and H^1 again must centralize Z_0', a contradiction. This proves (4.3).

(4.4). *One of the following holds*:

(1) $\bar{G}_0 \simeq L_3(2)$ *and* Z_0 *is the natural* $\bar{G}_0$*-module.*

(2) $Z_{d+1} \nleqslant M_1$ *for each* $\delta_{d+1} \in \Delta(\delta_d)$ *such that* $Z_0 \nleqslant G_{d+1}$.

Proof. Suppose (4.4) is false. Pick $\delta_{d+1} \in \Delta(\delta_d)$ *with* $Z_0 \nleqslant G_{d+1}$ but $Z_{d+1} \leqslant M_1$. Then by (4.3) one of the following holds:

(α) $|Z_{d+1}: Z_{d+1} \cap M_0| \geqslant |Z_0: Z_0 \cap M_{d+1}|$

(β) $|Z_0 \cap M_d: Z_0 \cap M_{d+1}| \geqslant |Z_{d+1}: Z_{d+1} \cap M_0|$.

As $\delta_0 \sim \delta_{d+1}$ we obtain that Z_0 is in any case an FF module for $\bar{G}_0$. Now by choice of δ_{d+1}

(*) $[Z_{d+1}, Z_0] > [Z_{d+1}, Z_0 \cap M_d] \leqslant Z_{d+1}$.

Suppose first (α) does not hold. Then $|Z_0: Z_0 \cap M_{d+1}| = 2\,|Z_{d+1}: Z_{d+1} \cap M_0|$. Since by (2.7) Z_0 and Z_{d+1} are both GF(q)-modules if $\bar{G}_0$ is defined over GF(q), this obviously implies $q=2$. Also by (2.7) visibly $\bar{G}_0$ is not isomorphic to $U_4(2)$, $U_5(2)$, $\Omega^-(6,2)$, or $G_2(2)$ since otherwise $|Z_{d+1}: Z_{d+1} \cap M_0| = |Z_0: C_{Z_0}(Z_{d+1})|$. If now $\bar{G}_0 \simeq \hat{A}_6$ or $\hat{\Sigma}_6$ and $|Z_0| = 2^6$, then $|Z_0: C_{Z_0}(Z_{d+1})| = 8$, which is impossible since Z_0 is a GF(4)-module. So we obtain that Z_0 is the natural Sp(4, 2)-module or the direct sum of two natural $L_3(2)$-modules. In any case $L_d = M_d\langle Z_0, M_{d+1}\rangle$ normalizes the hyperplane $Z_{d+1} \cap M_0$ of Z_{d+1}, which is impossible since L_d acts transitively on $\Delta(\delta_d)$.

Hence (α) holds and (*), (2.7), (2.8)(5), and (3.1) imply that $\bar{G}_0 \simeq A_6$, Σ_6 or $\hat{A}_6$, $\hat{\Sigma}_6$ and Z_0 is the natural module for one of these groups or that $\bar{G}_0 \simeq L_3(2)$ and Z_0 is the direct sum of two natural modules. Suppose first $\bar{G}_0 \simeq A_6$ or Σ_6. Then, arguing as above, $|Z_{d+1}: Z_{d+1} \cap M_0| = 4$ and $|[Z_{d+1}, Z_0 \cap M_d]| = 2$. Now let $F = C_{Z_{d+1}}(Z_0 \cap M_d)$. Then $[F, Z_0] = [Z_{d+1}, Z_0 \cap M_d] \leqslant Z_d$. Hence L_d normalizes F, a contradiction as above.

Next assume $\bar{G}_0 \simeq \hat{A}_6$ or $\hat{\Sigma}_6$ and $|Z_0| = 2^6$. Suppose $d > 3$. Then $R_{d+1} = \langle V_d^{G_{d+1}}\rangle$ is abelian and $R_{d+1} \leqslant C_{G_3}(Z_3)$. Since $\langle Z_0, M_{d+1}\rangle$ centralizes $Z_{d+1} \cap M_0$ and acts transitively on $\Delta(\delta_d)$, we obtain

$$L_d = \bigcap_{\mu \in \Delta(\delta_d)} Z_\mu = Z_{d+1} \cap M_0.$$

Thus $L_1 = [Z_0, Z_d] \leqslant V_d$ and $Z_2 = L_1 L_3 \leqslant C(R_{d+1})$. Hence $R_{d+1} \leqslant M_2$ and (2.6) implies that $|R_{d+1}: R_{d+1} \cap M_1| \leqslant 2$. Since $R_{d+1} \cap M_1$ acts quadratically on $Z_0 \cap M_d$ we have $R_{d+1} \cap M_1 \leqslant Z_{d+1} M_0$ and so $[R_{d+1} \cap M_1, Z_0 \cap M_d] \leqslant Z_{d+1}$. Now $Z_0 \cap M_d \leqslant M_{d+1} G'_{d+1}$ and $\hat{A}_6$ cannot act non-trivially on some GF(2)-module such that an involution induces a GF(2)-transvection. Hence $O^2(G_{d+1}) \leqslant C(R_{d+1}/Z_{d+1})$ and thus $R_{d+1} = V_d$, since $O^2(G_{d+1})$ acts already transitive on $\Delta(\delta_{d+1})$. But this contradicts $G = \langle G_d, G_{d+1} \rangle$.

Thus $d=3$. But then one obtains a contradiction to the fact that $[Z_0 \cap M_3, Z_4] \leqslant L_1 \cap L_3$ which is a 1-space in Z_2, considering Z_2 as a GF(4)-module, and to the action of $K_1/M_1 \simeq \Sigma_3$ on L_1.

If $\bar{G}_0 \simeq L_3(2)$ and $d=3$ then by assumption $Z_0 \cap M_3$ contains a natural $\bar{G}_0$-submodule Z'_0. Hence $[Z'_0, Z_4] \lhd G_1$ and so is contained in some natural $\bar{G}_2$-submodule of Z_2, a contradiction to $[Z'_0, Z_4]$ not normal in G_3. Thus $d>3$ and $Z_1 = [Z_0, Z_{d+1}] \leqslant V_d$. Hence $R_{d+1} \leqslant M_2$ and $|R_{d+1} : R_{d+1} \cap M_1| \leqslant 2$ by (2.6). Hence $[R_{d+1}, Z_0] \leqslant Z_1 V_d \leqslant V_d$, a contradiction to $Z_0 \nleqslant G_{d+1}$.

We fix for the rest of this section some $\delta_{d+1} \in \Delta(\delta_d)$ with $Z_0 \nleqslant G_{d+1}$. Then we have:

(4.5). *Either* $\bar{G}_0 \simeq L_3(2)$ *and* Z_0 *is the natural* $\bar{G}_0$*-module or* $Z_{d+1} \cap M_1 \nleqslant M_0$ *and* $Z_0 \cap M_d \nleqslant M_{d+1}$.

Proof. Assume (4.5) is false and with symmetry, that $Z_0 \cap M_d \leqslant M_{d+1}$. Since $L_1 = M_1 \langle M_0, Z_{d+1} \rangle$ acts transitively on $\Delta(\delta_1)$ this implies $Z_0 \cap M_d \lhd G_1$. Let Z'_0 be an irreducible $\bar{G}_0$-submodule of Z_0. Since $Z'_0 \cap Z_1$ is a non-trivial module for P_2, (3.1) and (2.10)(1) imply that either $Z'_0/[Z'_0, M_1]$ is the direct sum of non-trivial irreducible P_2-modules or $\bar{G}_0 \simeq L_3(2)$ and Z'_0 a natural module. (If $\bar{G}_0 \simeq L_3(4)$ then as Δ is non-spherical, $\bar{P}_0 \simeq L_2(4)$ or $L_2(4^3)$, a contradiction to $[Z_0, V_d \cap M_1] = 1$, since $V_d \leqslant M_2 \leqslant C(Z_0 \cap M_d)$.)

If in the first case $Z'_0 \nleqslant M_d$, then

$$[Z'_0, M_1] < (Z'_0 \cap M_d)[Z'_0, M_1] < Z'_0$$

by the above. But this is impossible since, as $K_1 \leqslant N(M_1 Z_{d+1})$, the middle group is normalized by K_1 and since $Z'_0/[Z'_0, M_1]$ is a direct sum of non-trivial irreducible K_1-modules. Hence $Z'_0 \leqslant Z_0 \cap M_d \leqslant M_{d+1}$ and $Z_{d+1} \leqslant C_{G_1}(Z'_0) = M_0$, a contradiction.

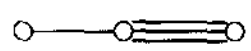
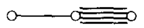
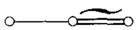
If in the second case $Z'_0 \leqslant M_d$ the same argument applies. Thus $Z_0 = (Z_0 \cap M_d) Z'_0$. Now, as $C_{Z_1}(P_2) = 1$, Z_1 is the direct sum of natural $L_2(2)$-modules. Since by assumption $Z_1 \nleqslant Z'_0$ this shows that Z_0/Z'_0 contains an irreducible $\bar{G}_0$-submodule V/Z'_0, is either equivalent to Z'_0 or the adjoint module. But then by [15, (2.2)] V is in any case the direct sum of two irreducible, whence by the above equivalent, $\bar{G}_0$-modules. But then as $|Z_0: Z_0 \cap M_d| = 2$, some $\bar{G}_0$-submodule of V must be contained in M_d, a contradiction as above.

(4.6). *Z_0 is an irreducible module for $\bar{G}_0$.*

Proof. Suppose (4.6) is false. Then by (4.3), (4.5), and (2.14) the hypothesis of (2.20) holds for Z_0 or Z_{d+1} and we may with symmetry assume it holds for Z_0.

By (3.1) and (2.20) $\bar{G}_0 \simeq \mathrm{Sp}(4, 2)$ or $L_3(2)$ and Z_0 contains the direct sum of two natural modules. Hence by (4.3) there exist irreducible submodules Z_0' of Z_0 and Z_{d+1}' of Z_{d+1} with $Z_0' \leqslant M_d$, $Z_{d+1}' \leqslant M_1$. By (4.5) $[Z_0', Z_{d+1}'] \neq 1$.

If $\bar{G}_0 \simeq \mathrm{Sp}(4, 2)$ then Z_0 is the direct sum of two natural modules. If $|Z_{d+1}': Z_{d+1}' \cap M_0| = 2$, then $|[Z_0', Z_{d+1}']| = 2$ and Z_0' centralizes a subgroup of index 4 in Z_{d+1}. Hence $Z_{d+1} \cap M_1 \leqslant Z_{d+1}' M_0$ and thus $L_d = M_d \langle Z_0, M_{d+1} \rangle$ normalizes a subgroup of index 4 in Z_{d+1}, a contradiction since L_d is transitive on $\Delta(\delta_d)$. Thus $|Z_{d+1}': Z_{d+1}' \cap M_0| = 4$ and $[Z_{d+1} \cap M_1, Z_0'] = [Z_{d+1}', Z_0'] \leqslant Z'_{d+1}$ which is impossible.

Thus $\bar{G}_0 \simeq L_3(2)$ and it is easy to see that Z_0 is the direct sum of two natural modules. Since Δ is non-spherical, $\Delta =$ , , or . Hence $G_0 \cap G_1$ centralizes Z_0/Z_1 and as $|V_1: V_1 \cap M_{d+1}| \leqslant 8$ it is easy to see that $G_1/M_1 \simeq \Sigma_3 \times \Sigma_3$. Then $|V_1/Z_1| = 2^4$ or 2^6 and in the latter case $1 \neq C_1/Z_1 = C_{V_1/Z_1}(G_1)$. Now $G_1 = K_1 L_1$, $C_{M_1}(V_1) = M_0 \cap M_1 \cap M_2 \cap M_\lambda$ for $\lambda \in \Delta(\delta_1)$, $\delta_0 \neq \lambda \neq \delta_2$. Thus $\hat{M}_1 = M_1/C_{M_1}(V_1)$ is the direct sum of two or three natural $K_1/M_1 \simeq \Sigma_3$-modules, which are permuted by $(L_1/M_1)' \simeq \mathbb{Z}_3$. Hence $C_{\hat{M}_1}(L_1') = 1$. But $[M_1, L_1'] \leqslant C_{M_1}(C_1)$ so that $M_1 \leqslant C(C_1)$. As $[K_1, L_1] \leqslant M_1$ this implies $L_1 \leqslant C(C_1)$ and C_1 is elementary abelian. But this contradicts $\Omega_1(Z(M_0 M_1)) = Z_1$.

This shows $V_1 = Z_0 Z_2$. Further $V_1' = \langle Z_0'^{G_1} \rangle = Z_0' Z_2'$, where $Z_2' \lhd G_2$, $|Z_2'| = 8$. Suppose

$$R_{d+1} = \langle Z_\lambda \mid \lambda \in \Delta_2(\delta_{d+1}) \rangle Z_{d+1} \leqslant M_2.$$

Then Z_0' induces a GF(2)-transvection on $\hat{R}_{d+1} = R_{d+1}/Z_{d+1}$ and so $\hat{R}_{d+1}/\hat{C}_{d+1}$ is a natural module dual to Z_{d+1}' for $\bar{G}_{d+1}$, where $\hat{C}_{d+1} = C_{\hat{R}_{d+1}}(O^2(G_{d+1}))$. Hence $Z_{d-1}' \leqslant C_{d+1}$ for some natural submodule of Z_{d-1} and setting $V_d' = \langle Z_{d-1}'^{G_{d-1}} \rangle$ we obtain $Z_{d+1} V_d' \lhd G_{d+1}$. Now since $Z_1' \leqslant C(V_d')$ and $Z_{d-1}' \leqslant C(Z_0')$, it follows that $|[V_d', Z_0']| = 2$. Hence, if H_{d+1} is a subgroup of G_{d+1} generated by three conjugates of Z_0' with $M_{d+1} H_{d+1} = G_{d+1}$, then $Z_{d+1} V_d' = Z_{d+1} C_{Z_{d+1} V_d'}(H_{d+1})$. Since $G_d \cap G_{d+1}$ centralizes V_d'/Z_{d+1}' this shows $V_d' \lhd G_{d+1}$, a contradiction.

Thus $R_{d+1} \nleqslant M_2$ and $R_0 \nleqslant M_{d-1}$. Suppose $d > 3$ and pick $\delta_{-1} \in \Delta(\delta_0)$

with $Z_{-1} \cap [Z'_0, Z'_{d+1}] = 1$. Claim $V_{-1} \leqslant M_{d-1}$. Namely if $(\delta_{-1}, \delta_{d-1})$ is a critical pair, then

$$1 \neq [Z'_{-2}, Z'_{d-1}] \leqslant C_{Z'_{-1}}(Z'_{d+1}) \leqslant Z'_1,$$

where $Z'_{-1} = Z_{-1} \cap Z'_0$, $\delta_{-2} \in \Delta(\delta_{-1})$ with $Z_{d-1} \nleqslant G_{-2}$, and Z'_{-2} is the irreducible submodule of Z_{-2} containing Z'_{-1}. But then $Z'_1 = [Z'_0, Z'_{d+1}]$ $[Z'_{-2}, Z'_{d-1}] \leqslant R_{d+1}$, a contradiction to $R_{d+1} \nleqslant M_2$. Hence $(\delta_{-1}, \delta_{d-1})$ is not a critical pair and $[V'_{-1}, Z_{d-1}] \leqslant Z_{d-1} \cap Z'_{-1} = 1$ by the same reason. Thus $V_{-1} = Z_0(V_{-1} \cap M_d)$ and, since $R_0 \nleqslant M_{d-1}$,

$$[V_{-1} \cap M_d, Z'_{d+1}] = [Z'_0, Z'_{d+1}] \leqslant Z'_0.$$

But then $G_0 = \langle G_0 \cap G_{-1}, Z'_{d+1} \rangle \leqslant N(V_{-1})$, again a contradiction.

Hence $d = 3$. But then, as $V'_1 = Z'_0 \cdot Z'_2 \lhd G_1$, we have $[Z'_0, Z_4] \leqslant Z'_2 \cap Z_3$ and so $|[Z'_0, Z_4]| = 2$. This final contradiction proves (4.6).

Set for the rest of Section 4 $F_1 = \bigcap Z_\lambda$, $\lambda \in \Delta(\delta_1)$ and similarly F_μ for $\mu \in \Gamma$ with $\mu \sim \delta_1$.

(4.7). *One of the following holds*:

(1) $\bar{G}_0 \simeq L_3(2)$ *and* Z_0 *is the natural* $\bar{G}_0$-*module.*

(2) $G_0/C_{G_0}(Z_0) \simeq \mathrm{Sp}(4, 2)$ *or* A_6 *and* Z_0 *is the natural module.*

(3) $\bar{G}_0 \simeq \Omega(6, 2)$ *and* Z_0 *is the natural* $\bar{G}_0$-*module.*

(4) $\bar{G}_0 \simeq G_2(2)$ *or* $G_2(2)'$ *and* $|Z_0| = 2^6$.

(5) $\bar{G}_0 \simeq \hat{A}_6$ *or* $\hat{\Sigma}_6$, $|Z_0| = 2^6$ *and* Z_0 *is the* $\bar{G}_0$-*module obtained from* $\hat{A}_6 \subsetapprox \mathrm{SL}_3(4)$.

Proof. Suppose (4.7) is false. By (4.3)–(4.6) and (2.5)(2), (2.14) one of the cases of (2.18) holds. Then, as $\langle M_0, Z_{d+1} \rangle$ acts transitively on $\Delta(\delta_1)$, $[Z_0 \cap M_d, Z_{d+1} \cap M_1] \leqslant F_1 \cap F_d$. Now by (2.19) and (3.1)

$$(*) \quad |[Z_0 \cap M_d, Z_{d+1}] : [Z_0 \cap M_d, Z_{d+1} \cap M_1]| \leqslant 2,$$

since we assume (4.7) does not hold. Hence, if $x \in Z_{d+1} - (Z_{d+1} \cap M_1)$, then $|[Z_0 \cap M_d/F_1, x]| \leqslant 2$. But on the other hand $C_{Z_0/F_1}(x) = 1$, since $\langle M_0, x \rangle$ is already transitive on $\Delta(\delta_1)$. Hence $|Z_0 : F_1| \leqslant 4$ and it is obvious with (2.18) and (3.1) that (4.7) holds.

(4.8). *Suppose* $d > 3$. *Then* $R_{d+1} \cap M_1 M_2 \leqslant Z_{d+1} M_1$ *and* $R_0 \cap M_d M_{d-1} \leqslant Z_0 M_d$.

Proof. $R_{d+1} \cap M_1 M_2$ centralizes $[V_1, Z_{d+1}] \leqslant V_d$. But if $P_0 Q_0$ is not defined over GF(2), then (2.6) implies that V_1/F_1 contains a Steinberg module for L_1/M_1, $L_1 = M_1 \langle M_0, Z_{d+1} \rangle$ and so does not admit a quadratically acting four-group.

(4.9). *$\bar{G}_0$ is not isomorphic to $\Omega^-(6, 2)$, $G_2(2)$, or $G_2(2)'$.*

Proof. Suppose (4.9) is false. Then $|Z_0| = 2^6$ by (4.7). Claim

$$(*) \quad |Z_{d+1} \cap M_1 : Z_{d+1} \cap M_0| = 2.$$

If not, then $|Z_{d+1} \cap M_1 : Z_{d+1} \cap M_0| \geqslant 4$ and $|C_{Z_0}(Z_{d+1} \cap M_1)| \leqslant 8$. If $C_{Z_0}(Z_{d+1}) > Z_1$, then as $C_{Z_0}(Z_{d+1})$ is $\langle M_0, Z_{d+1}\rangle$ invariant, it follows $|C_{Z_0}(Z_{d+1})| = 2^4$, since $\langle M_0, Z_{d+1}\rangle$ is transitive on $\Delta(\delta_1)$. Thus $C_{Z_0}(Z_{d+1}) = Z_1$, $C_{Z_{d+1}}(Z_0) = Z_d$, and $|Z_{d+1} \cap M_1 : Z_{d+1} \cap M_0| = 8$, a contradiction to $[Z_{d+1} \cap M_1, Z_0] \leqslant C_{Z_0}(Z_{d+1})$.

Thus $(*)$ holds and $C_{Z_0}(Z_{d+1}) = F_1$, $C_{Z_{d+1}}(Z_0) = F_d$. Pick $\lambda \in \Delta(\delta_{d+1})$ with $Z_\lambda \cap F_d = 1$. Then by (4.3) $|V_\lambda : V_\lambda \cap M_3| \leqslant 2 \geqslant |Z_2 : Z_2 \cap M_\lambda|$ and

$$[Z_2 \cap M_\lambda, V_\lambda \cap M_3] \leqslant Z_2 \cap Z_\lambda \leqslant C_{Z_{d+1}}(Z_0) \cap Z_\lambda = 1$$

since Z_2 centralizes $F_\lambda \leqslant Z_{d+1}$ and $|Z_\mu : F_\lambda| = 4$ for each $\mu \in \Delta(\lambda)$. Hence $Z_2 \cap M_\lambda$ centralizes a hyperplane in each Z_μ, $\mu \in \Delta(\lambda)$ and thus $[Z_2 \cap M_\lambda, V_\lambda] = 1$. But then $V_\lambda \leqslant M_2$. If now $d > 3$ then by (4.8) $V_\lambda = Z_{d+1}(V_\lambda \cap M_1)$ and, since V_λ centralizes F_1, $[V_\lambda, Z_0 \cap M_d] \leqslant Z_{d+1}[Z_{d+1} \cap M_1, Z_0 \cap M_d] \leqslant Z_{d+1} \leqslant V_\lambda$. But as $C_{Z_{d+1}}(Z_0 \cap M_d) = F_d$ we have $Z_0 \cap M_d \nleqslant G_\lambda$, a contradiction.

Hence $d = 3$ and $[Z_0, V_3 \cap M_1] \leqslant F_1 \leqslant Z_2$. But then $[Z_0, V_3] \leqslant Z_2$ by the action of $O^2(G_2 \cap G_3)$ on V_3/F_3. This easily shows that $V_3 = Z_2 Z_4$ and $V_1 = Z_0 Z_2$. Hence $[V_1, Z_4] \leqslant Z_2 \leqslant C_{V_1}(V_\lambda)$ even in this case, and the argument of (4.8) shows $V_\lambda = Z_4(V_\lambda \cap M_1)$. But then we obtain a contradiction as in case $d > 3$.

(4.10). $G_0/C_{G_0}(Z_0) \not\simeq \hat{A}_6, \hat{\Sigma}_6$.

Proof. Suppose (4.10) is false. Then by (4.8) $|Z_0| = 2^6$. Since $|Z_0 \cap M_{d+1}| \geqslant 8$ and $\langle M_0, Z_{d+1}\rangle$ is transitive on $\Delta(\delta_1)$, it follows that $|F_1| = 2^4 = |F_d|$. Further $|F_1 \cap F_3| = 4$ since both are 2-dimensional subspaces of Z_2, considering Z_2 as a GF(4)-module. Claim

$$(*) \quad R_{d+1} \nleqslant M_2 \qquad \text{and} \qquad R_0 \nleqslant M_{d-1} \qquad \text{if} \quad d > 3.$$

Suppose $(*)$ is false for R_{d+1}. Then $R_{d+1} = Z_{d+1}(R_{d+1} \cap M_1)$ by (4.8) and $|R_{d+1} \cap M_1 : R_{d+1} \cap M_0| \leqslant 4$ since R_{d+1} centralizes F_1. Hence $Z_0 \cap M_d$ centralizes a GF(2)-hyperplane of R_{d+1}/Z_{d+1} and thus, since $Z_0 \cap M_d \leqslant M_{d+1} G'_{d+1}$, $[O^2(G_{d+1}), R_{d+1}] \leqslant Z_{d+1}$ (since an involution of $\hat{A}_6$ cannot act as a GF(2)-transvection!). But then $R_{d+1} = V_d$, a contradiction. This proves $(*)$.

Next we show $[Z_0 \cap M_d, Z_{d+1} \cap M_1] \leqslant F_3$. If $d = 3$ this is obvious. If $d > 3$ and the statement is false, then R_{d+1} centralizes a hyperplane of Z_2, whence $R_{d+1} \leqslant M_2$, a contradiction to $(*)$. Since $F_1 \cap F_3$ is a 1-space of Z_0

we obtain $[Z_0, Z_{d+1} \cap M_1] = F_1 \cap F_3 = C_{Z_0}(R_{d+1})$. Pick $\delta_{-1} \in \Delta(\delta_0)$ with $F_{-1} \cap F_1 \cap F_3 = 1$. Then, if $d > 3$,

$$[V_{-1} \cap M_{d-2}, Z_{d-1} \cap M_{-1}] \leqslant F_{-1} \cap Z_{d-1} \leqslant F_{-1} \cap C_{Z_0}(R_{d+1}) = 1.$$

Hence (4.3) implies $[V_{-1}, Z_{d-1} \cap M_{-1}] = 1$ and so by (4.5) $(\delta_{-1}, \delta_{d-1})$ is not a critical pair. But then, again by (4.5), $Z_{d-1} \leqslant M_{-1}$ and $V_{-1} \leqslant M_{d-2}$, whence as before $[V_{-1}, Z_{d-1}] = 1$. Thus by (4.8) $V_{-1} = Z_0(V_{-1} \cap M_d)$ and by (*)

$$[V_{-1} \cap M_d, Z_{d+1} \cap M_1] \leqslant C_{Z_{d+1}}(R_0) \leqslant F_d \cap F_{d-2}.$$

Hence $V_{-1} \cap M_d \leqslant (Z_0 \cap M_d) M_{d+1}$ and so

$$[V_{-1}, Z_{d+1} \cap M_1] \leqslant Z_0[Z_0 \cap M_d, Z_{d+1} \cap M_1] \leqslant Z_0 \leqslant V_{-1},$$

a contradiction to $Z_{d+1} \cap M_1 \nleqslant G_{-1}$.

This shows $d = 3$. But then $[Z_0, Z_4 \cap M_1] = F_1 \cap F_3 \leqslant Z_4$. Hence $L_3 = M_3\langle Z_0, M_4\rangle$ normalizes $Z_4 \cap M_1$, which is impossible.

(4.11). *Suppose $G_0/C_{G_0}(Z_0) \simeq \Sigma_6$ or A_6. Then the following hold*:

(1) $|Z_{d+1} \cap M_1 : Z_{d+1} \cap M_0| = 2$ *and* $Z_{d+1} \cap M_1$ *induces a* GF(2)-*transvection on* Z_0.

(2) $G_0/C_{G_0}(Z_0) \simeq \Sigma_6$.

Proof. It is obvious that (2) is a consequence of (1). So assume (1) is false. As $Z_d \leqslant M_0$ in any case by (4.7) $|Z_{d+1} \cap M_1 : Z_{d+1} \cap M_0| = 2$, but

$$Z_1 = [Z_{d+1} \cap M_1, Z_0] \leqslant V_d \leqslant R_{d+1}.$$

Thus if $d > 3$, $R_{d+1} \leqslant M_2 M_1$ and by (4.8) $R_{d+1} = Z_{d+1}(R_{d+1} \cap M_1)$. Since $[Z_0, Z_{d+1}] \leqslant V_d \leqslant R_{d+1}$ this implies $Z_0 \leqslant N_{G_d}(R_{d+1}) = G_d \cap G_{d+1}$, a contradiction to (4.5).

Thus $d = 3$, $Z_1 \cap Z_3 \geqslant [Z_4 \cap M_1, Z_0 \cap M_3] \neq 1$ and $Z_1 = C_{Z_0}(Z_4 \cap M_1)$, $Z_3 = C_{Z_4}(Z_0 \cap M_3)$. Since $[Z_0, Z_4 \cap M_1] = Z_1 \leqslant Z_2$, the action of $G_2 \cap G_3$ on V_3/Z_3 implies $[Z_0, Z_4] \leqslant Z_2$. Hence easily $V_1 = Z_0 Z_2$ and $L_1/M_1 \simeq \Sigma_3$, where $L_1 = M_1\langle M_0, Z_4\rangle$.

Pick now $\delta_5 \in \Delta(\delta_4)$ with $Z_3 \cap Z_5 = 1$. Then (δ_2, δ_5) is not a critical pair, since otherwise by (4.5) $1 \neq [Z_2 \cap M_5, V_5 \cap M_3] \leqslant Z_3 \cap Z_5$. Hence

$$[Z_2, V_5] \leqslant Z_3 \cap Z_5 = 1.$$

Hence $V_5 = Z_4(V_5 \cap M_1)$ and either $V_5 \cap M_0 > Z_3$ or $|V_5 \cap M_1 : V_5 \cap M_0| = 8$. In any case there exists an $x \in V_5 \cap M_1 - Z_4$ centralizing $Z_0 \cap M_3$. Let $H_4 = \langle Z_0 \cap M_3, M_5\rangle$. Then, as $M_0 M_1 = Q_2$, either $M_4 H_4 = G_4$ or

$M_4H_4/M_4 \simeq A_5$. In any case δ_3 and δ_5 are interchanged by H_4. Hence $V_3 \cap V_5 > Z_4$ and, since $G_3 \cap G_4 \cap G_5$ acts irreducibly on V_3/Z_4 and V_5Z_4, we obtain $V_3 = V_5$. This contradiction proves (4.11).

(4.12). *If* $\bar{G}_0 \simeq \Sigma_6$ *then* $d=3$.

Proof. Suppose (4.12) is false. Then $Z_{d+1} \cap M_1$ induces by (4.11) a transvection on Z_0. Claim

$$(*)\quad Z_1 \nleqslant R_{d+1} \qquad \text{and} \qquad Z_d \nleqslant R_0.$$

Suppose $Z_d \leqslant R_0$. Then $R_0 \leqslant M_{d-1}M_d$ since it centralizes Z_dZ_{d-2}. By (4.8) $R_0 = Z_0(R_0 \cap M_d)$ and thus

$$[R_0, Z_{d+1}] \leqslant Z_d[Z_0, Z_{d+1}] \leqslant Z_dV_1 \leqslant R_0,$$

a contradiction to $Z_{d+1} \nleqslant G_0$ by (4.5).

Pick now $\delta_{-1} \in \Delta(\delta_0)$ such that $Z_{-1} \cap Z_1 = 1$. Then $[V_{-1}, Z_{d-1} \cap M_{-1}] \leqslant Z_{-1} \cap V_{d-2} \leqslant Z_{-1} \cap R_{d-1} \leqslant Z_{-1} \cap C_{Z_0}(Z_{d+1}) = 1$, since obviously $Z_1 = C_{Z_0}(Z_{d+1})$. Hence $(\delta_{-1}, \delta_{d+1})$ is not a critical pair and (4.3), (4.5) imply

$$[V_{-1}, Z_{d-1}] \leqslant Z_{-1} \cap Z_{d-1} = 1.$$

Hence by (4.8) $V_{-1} = Z_0(V_{-1} \cap M_d)$ and by $(*)$ $|V_{-1} \cap M_d : V_{-1} \cap M_{d+1}| \leqslant 4$. Let $A = (V_{-1} \cap M_{d+1})Z_0$. Then, as shown, $|V_{-1} : A| \leqslant 2$ and $H_0 = \langle M_{-1}, Z_{d+1} \cap M_1 \rangle$ normalizes A. But as in (4.11) $M_0H_0 = G_0$ and $H_0 \cap G_{-1} \cap G_{-2}$ acts irreducibly on Z_{-2}/Z_{-1} for each $\delta_{-2} \in \Delta(\delta_{-1})$. Hence $V_{-1} = A = V_1$ which is obviously impossible.

Now we can finally show:

(4.13). *One of the following holds*:

(1) $d=1$ *or*

(2) $\bar{G}_0 \simeq L_3(2)$ *and* Z_0 *is the natural* $\bar{G}_0$*-module.*

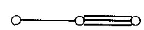
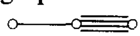
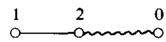
Proof. By (4.6)–(4.12) we only need to show that $d>3$ if $G_0/C_{G_0}(Z_0) \simeq \Sigma_6$ and $|Z_0| = 2^4$. So assume $d=3$ in this case. Suppose first $Z_1 \cap Z_3 = 1$. Then there exists a subgroup $\mathbb{Z}_3 \simeq K \leqslant K_1 \cap G_2 \cap K_3$ acting irreducibly on Z_1 and Z_3. But then $Z_0 = \langle (Z_0 \cap M_3)^K \rangle \leqslant M_3$, a contradiction.

Hence $Z_1 \cap Z_3 \simeq \mathbb{Z}_2$ and if $\delta_{-1} \in \Delta(\delta_0)$ such that $Z_{-1} \cap Z_1 = 1$, then (δ_{-1}, δ_2) is not a critical pair. But then by (4.4)

$$[V_{-1}, Z_2] \leqslant Z_{-1} \cap Z_1 = 1$$

for such a δ_{-1}. Now as $[Z_0, Z_4 \cap M_1] \leqslant Z_2$ we obtain as in (4.11) $[Z_0, Z_4] \leqslant Z_2$ and $V_3 = Z_2 Z_4$. Hence $P_0/Q_0 \simeq \Sigma_3$ and $G_1/M_1 \simeq \Sigma_3 \times \Sigma_3$. Thus $V_{-1} = Z_0(V_{-1} \cap M_3)$ and $[V_{-1}, Z_4 \cap M_1] \leqslant Z_0 Z_3 \leqslant V_1$. Now by the structure of Σ_6 there exists a subgroup $\mathbb{Z}_3 \simeq R \leqslant G_{-1} \cap G_0 \cap G_1$ satisfying $Q_2 = M_0 \langle (Z_4 \cap M_1)^R \rangle$. Hence $[V_{-1}, M_1] \leqslant V_1$ and with symmetry $[V_1, M_{-1}] \leqslant V_{-1}$. But then $V_1 V_{-1} \lhd G_0 = M_0 \langle M_1, M_{-1} \rangle$, a contradiction to $R_0' \neq 1$.

5. A Non-existence Theorem

In this section we carry on with the hypothesis and notation introduced in Section 3. We assume in addition that $\bar{G}_0 \simeq L_3(q)$ and $\bar{G}_2 \not\simeq \hat{A}_6, \hat{\Sigma}_6$. By (3.8) and (4.13) either $d \leqslant 2$ or $\bar{G}_0 \simeq L_3(2)$ and Z_0 is the natural $\bar{G}_0$-module. Assume for the rest of this section that the second possibility holds. Then $Z = Z_2 = \langle Z^{G_2} \rangle \simeq \mathbb{Z}_2$ and $|Z_1| = 4$. We consider in this section the coset-graph $\Gamma = \Gamma(G_0, G_2)$. Since Δ is non-spherical, $\Delta =$  or  and $\bar{G}_2 \in \{G_2(2)', G_2(2), {}^3D_4(2), {}^2F_4(2), \text{ or } {}^2F_4(2)'\}$. To simplify notation we change the enumeration of Δ to $\Delta =$ . (I.e., we interchange G_1 and G_2!).

Use the same notation as in Section 3. Choose for the rest of this section a fixed vertex δ_d such that (δ_0, δ_d) is a critical pair, where $\delta_0 = G_0$. Let $\delta_0, \delta_1, ..., \delta_{d-1}, \delta_d$ be a fixed arc from δ_0 to δ_d and assume without loss of generality $\delta_1 = G_1$. As in Section 3 we drop the δ for groups corresponding to vertices of this arc.

We first show:

(5.1). *Suppose $d \equiv 0$ (2). Then the following hold:*

(1) $d > 4$.

(2) $\bar{G}_1 \simeq G_2(2)$ *or* $G_2(2)'$.

(3) V_1/Z_1 *is the natural* $\bar{G}_1$*-module or a non-split extension of the natural* $\bar{G}_1$*-module by a trivial.*

Proof. Suppose $d = 2$ and pick $\delta_3 \in \Delta(\delta_2)$ such that $V_1 \cap G_3 \nleqslant M_2 M_3 = O_2((G_2 \cap G_3)$. Then $\overline{(V_1 \cap G_3)} \bar{M}_2 = \bar{T} \in \mathrm{Syl}_2(\bar{G}_3)$ and $\overline{V_1 \cap G_3}$ is an elementary abelian normal subgroup of $\bar{T}$ not contained in $\bar{M}_2$, a contradiction to (2.13). Next suppose $d = 4$. Then $Z_1 = [Z_0, Z_4] = Z_3 \lhd G_2$, which is impossible. This proves (1).

Next suppose (2) is false. Pick $\delta_{d+1} \in \Delta(\delta_d)$ with $Z_0 \cap G_{d+1} \nleqslant M_d M_{d+1}$. As $Z_1 = Z_{d-1}$ (1) implies $V_{d+1} \leqslant G_1$. Hence by (2.14) $|V_{d+1} : V_{d+1} \cap M_1| \leqslant 2^5$ if $\bar{G}_1 \simeq {}^3D_4(2)$ resp. $\leqslant 2^6$ if $\bar{G}_1 \simeq {}^2F_4(2)$ or ${}^2F_4(2)'$. Hence $Z_0 \cap G_{d+1}$

centralizes a subspace of index $\leqslant 2^6$ resp. 2^7 in V_{d+1}/Z_{d+1}, a contradiction to (2.14) since Z_0/Z_1 is a natural submodule of V_1/Z_1 of $\widetilde{G_0 \cap G_1}/M_0 M_1 \simeq \Sigma_3$.

Now with (2.15) the same argument shows that V_1/Z_1 contains only natural and trivial $\bar{G}_1$-composition factors. Suppose that V_1/Z_1 contains two natural composition factors and let δ_{d+1} be as before. Then $|V_1 : V_1 \cap G_{d+1}| \leqslant 2$, $|V_{d+1} : V_{d+1} \cap M_1| \leqslant 8$, and $[V_{d+1} \cap M_1, V_1 \cap G_{d+1}] \leqslant Z_1 = Z_{d-1}$. Applying (2.15) to the action of $\bar{G}_{d+1}$ on V_{d+1}/Z_{d+1} this implies $|V_1 \cap G_{d+1} : V_1 \cap M_{d+1}| = 2$, a contradiction to $|[V_{d+1}, V_1 \cap M_{d+1}]| \leqslant 2$ and $V_{d+1} \not\leqslant M_1$.

As $V_1/Z_1 = \langle (Z_0/Z_1)^{G_1} \rangle$, this shows that V_1/Z_1 is the extension of a natural by a trivial $\bar{G}_1$-module. Further, since $Z_0/Z_1 \leqslant [V_1/Z_1, G_1]$, the extension does not split.

(5.2). *V_1/Z_1 is the natural $\bar{G}_1$-module if $d \equiv 0$ (2).*

Proof. Suppose (5.2) is false and pick $\delta_{d+1} \in \Delta(\delta_d)$ such that $V_1 \cap G_{d+1} \not\leqslant M_d M_{d+1}$. Then $V_{d+1} \leqslant G_1$ since $Z_1 = [Z_0, Z_d] = Z_{d-1}$.

Let $C_1/Z_1 = C_{V_1/Z_1}(\bar{G}_1)$. Then $|C_1| = 4$ and $[C_1, G_1] = Z_1$ by (5.1). Claim $C_1 Z_0 \ntrianglelefteq G_0$. If our claim is false, then $M_0 \leqslant C_{G_0}(C_1 Z_0)$, since $|M_0 : M_0 \cap C_{G_1}(C_1)| \leqslant 2$. But, as $[C_1 Z_0, Z_d] \leqslant Z_1$, Z_d induces a GF(2)-transvection on $C_1 Z_0$ so that $C_1 Z_0 = Z_0 C_{C_1 Z_0}(G_0)$, a contradiction.

Let $R_0 = \langle V_\lambda \mid \lambda \in \Delta(\delta_0) \rangle$ and $\tilde{R}_0 = R_0/Z_0$. Then by (5.1)(1) R_0 is elementary abelian and $\tilde{R}_0$ has at least three non-trivial $\bar{G}_0$-composition factors, one of which is not the dual module of Z_0, since $\tilde{V}_1$ has the $G_0 \cap G_1$ composition series

$$\tilde{V}_1 > [\tilde{V}_1, M_0] > \tilde{C}_1 > 1$$

and the first two composition factors are non-trivial $G_0 \cap G_1$-modules. (If $\tilde{R}_0$ contains a Steinberg submodule $\tilde{W}$, then $\tilde{W} \cap \tilde{C}_1 = 1$, a contradiction to $\tilde{C}_1 \leqslant [\tilde{V}_1, M_0]$!)

But on the other hand $R_0 \leqslant G_{d-1}$ and thus $|R_0 : R_0 \cap M_{d-1}| \leqslant 8$. Since $[R_0 \cap M_{d-1}, V_{d-1}] \leqslant Z_{d-1} \leqslant Z_0$, this implies $|V_{d-1} : V_{d-1} \cap M_0| \leqslant 2$. Now by symmetry also $|V_1 : V_1 \cap M_d| = 2$ and thus $V_1 \leqslant G_{d+1}$. Obviously $|V_{d+1} : V_{d+1} \cap M_1| \geqslant 4$ since $|[V_{d+1}, t]| \geqslant 8$ for $t \in V_1 - M_d M_{d+1}$ and thus $Z_{d+1} = [V_1 \cap M_{d+1}, V_{d+1}]$, since V_{d+1} cannot centralize a subgroup of index 4 in V_1. ($|V_1 : V_1 \cap M_{d+1}| \leqslant 4$, since $V_1 \not\leqslant M_d M_{d+1}$ and acts quadratically on V_{d+1}!) Now, by the structure of $U_3(3)$, all involutions of $(G_d \cap G'_{d+1}) M_{d+1}/M_{d+1}$ lie in $M_d M_{d+1}/M_{d+1}$. Since $V_1 \not\leqslant M_d M_{d+1}$ by choice, and since $|V_1 : V_1 \cap M_d| = 2$ and $|V_1 : V_1 \cap M_{d+1}| = 4$, this implies $V_1 \cap M_d = V_1 \cap M_{d+1} G'_{d+1}$. Hence $|[V_1 \cap M_d, V_{d+1}/Z_{d+1}]| = 4$. But V_{d+1} cannot centralize a subgroup of index 4 in V_1 so that $[V_1 \cap M_{d+1}, V_{d+1}] = Z_{d+1}$ and $|[V_1 \cap M_d, V_{d+1}]| = 8$. Further, as

$Z_{d+1} \leqslant V_1$, we have $C_{Z_d}(V_1) = Z_{d-1}Z_{d+1} \leqslant [V_1 \cap M_d, V_{d+1}]$. Hence $|[V_1 \cap M_d/Z_1, V_{d+1}]| = 4$.

Now, as R_d/Z_d contains at least three non-trivial $\tilde{G}_d$-composition factors, $|R_d: R_d \cap M_1| > 4$. Hence $R_d \leqslant M_1 M_2$, since R_d acts quadratically on V_1. Therefore (2.16) implies

$$Z_2 \leqslant [V_1, R_d] = [V_1, V_{d+1}] \leqslant V_1 \cap V_{d+1}.$$

Let $F_{d+1} = \langle R_d^{G_{d+1}} \rangle$. Then $[V_{d+1}, F_{d+1}] = 1$ since $d > 4$. Hence $F_{d+1} \leqslant M_2$ and centralizes $[V_1, R_d]$. This implies $F_{d+1} \leqslant M_1 R_d$ and thus $[V_1, F_{d+1}] \leqslant V_{d+1}$. This shows that F_{d+1}/V_{d+1} is a trivial module for $\langle V_1^{G_{d+1}} \rangle$ and so

$$F_{d+1} = R_d \lhd \langle G_d, G_{d+1} \rangle = G,$$

a contradiction.

(5.3). $d \equiv 1$ (2).

Proof. Suppose false and pick $\delta_{d+1} \in \Delta(\delta_d)$ with $V_1 \cap G_{d+1} \nleqslant M_d M_{d+1}$. Claim

(*) $\hat{R}_d = R_d/Z_d$ has exactly two G_d-composition factors, which are both natural $\tilde{G}_d$-modules dual to Z_d.

Since $\hat{V}_{d+1}/[\hat{V}_{d+1}, M_d]$ and $[\hat{V}_{d+1}, M_d]$ are both natural $L_2(2)$-modules for $G_d \cap G_{d+1}$ and since $[\hat{R}_d, M_d] = \langle [\hat{V}_{d+1}, M_d]^{G_d} \rangle$ (*) is obvious if we can show that $\hat{R}_d/[\hat{R}_d, M_d]$ and $[\hat{R}_d, M_d]$ are both FF modules. If $|R_d: R_d \cap M_1| \leqslant 4$ or if $|V_1 : V_1 \cap M_d| \geqslant 4$ this is the case. So we may assume $|V_1 : V_1 \cap M_d| = 2$, whence $V_1 \leqslant G_{d+1}$ and $R_d \leqslant M_1 M_2$ since it acts quadratically on V_1. But then, as $|V_{d+1}: V_{d+1} \cap M_1| \geqslant 4$ since $V_1 \nleqslant M_d M_{d+1}$, we obtain

$$Z_2 \leqslant [V_1, V_{d+1}] \leqslant V_1 \cap V_{d+1}$$

and thus $F_{d+1} \leqslant M_2$ and $[V_1, F_{d+1}] = [V_1, V_{d+1}] \leqslant V_{d+1}$ as in (5.2). Hence again $F_{d+1} = R_d$, a contradiction.

Now (*) implies $|R_d: R_d \cap M_1| = 4$ (as $V_{d-1} \leqslant M_1$!). Since by (*) applied to R_2, $|V_1 : V_1 \cap V_3| \leqslant 4$ and since $|V_1 \cap G_{d+1} : V_1 \cap M_{d+1}| \geqslant 4$, we have $V_1 \leqslant G_{d+1}$. We show that this is impossible.

Pick $\lambda \in \Delta(\delta_d)$ with $V_1 \nleqslant G_\lambda$. Then, as $V_3 \leqslant M_\lambda$ and $|V_1 : V_1 \cap M_d| = 2$, since $Z_{d-1}Z_{d+1} = C_{Z_d}(V_1)$, we obtain $|V_1 \cap M_d: V_1 \cap M_\lambda| \leqslant 2$. Now $[V_1 \cap M_\lambda, V_\lambda] \leqslant V_1 \cap Z_\lambda = 1$. Hence V_λ centralizes a subgroup of index at most 4 in V_1 and so $|V_\lambda : V_\lambda \cap M_1| \leqslant 2$, a contradiction to $R_d = V_{d-1} V_\lambda$ by (*). This proves (5.3).

(5.4). *The following hold*:

(1) $d>1$

(2) $\bar{G}_1 \simeq G_2(2)$ *or* $G_2(2)'$.

Proof. If (1) is false then $[Z_0, M_1] \leqslant Z_1$ and $Z_0 \nleqslant M_1$. Hence $O^2(\tilde{G}_1)$ centralizes M_1, a contradiction to hypothesis (3) of Section 3. Assume (2) is false. Then by (2.14) $|V_d: V_d \cap M_1| \leqslant 2^6$ resp. 2^5 in case of ${}^3D_4(2)$. Further $[V_d \cap M_1, V_1] \leqslant Z_1$, so that each element of $V_1 M_d/M_d$ centralizes a subgroup of index 2^7 resp. 2^6 in V_d/Z_d. Hence (2.14) shows $\bar{G}_1 \simeq {}^3D_4(2)$ and all elements of $\bar{V}_1^{\#}$ are long root elements of $\bar{G}_d$. Hence $|V_1: V_1 \cap M_d| \leqslant 4$ and thus every element of $V_d M_1/M_1$ centralizes a subgroup of index at most 8 of V_1/Z_1. This implies $V_d \leqslant M_1$, a contradiction to $V_1 \nleqslant M_d$.

(5.5). (a) *V_1/Z_1 is the natural $\bar{G}_1$-module or an indecomposable module, which is the extension of a natural by a trivial module.*

(b) *If $C_1/Z_1 = C_{V_1/Z_1}(\bar{G}_1) \neq 1$, then $Z_0 C_1$ is not normal in G_0.*

Proof. The proof of (a) is exactly the same as the proof of (5.1)(3). To prove (b) note that $M_0 O^2(G_0 \cap G_1) \leqslant C(C_1)$ if $Z_0 C_1 \lhd G_0$, since $[Z_0 C_1, M_0] \leqslant Z_1$. Hence the elements of $M_1 - M_0$ act as GF(2)-transvections on $Z_0 C_1$, so that $Z_0 C_1 = Z_0 \times C_{Z_0 C_1}(G_0)$.

(5.6). $d>3$.

Proof. Suppose $d=3$. We first show

(∗) $[V_1, V_3] = Z_2$ *and* $|V_3: V_3 \cap M_1| = 2$.

Let $T = G_1 \cap G_2 \cap G_3$. Then $M_1 T \in \mathrm{Syl}_2(G_1)$ and $M_1 V_3$ is $M_1 T$-invariant. Hence, if $|V_3: V_3 \cap M_1| = 2$, then obviously (∗) holds.

Next suppose $|V_3: V_3 \cap M_1| = 4$. Then all elements of $V_3 M_1/M_1$ are long root involutions of $\bar{G}_1$ by the structure of a 2-Sylow subgroup of $G_2(2)$. Hence by (2.16) there exists a hyperplane H of V_1 such that $|[H/Z_1, V_3|] = 2$, whence $|[H, V_3]| \leqslant 4$. On the other hand, $H \nleqslant M_3$, as G_2 is 2-transitive on $\Delta(\delta_2)$. Hence $|[H, V_3/Z_3]| \geqslant 4$ and thus $|[H, V_3]| \geqslant 8$, since $Z_3 = [H \cap M_3, V_3]$ if $|H: H \cap M_3| = 2$.

We obtain $|V_3: V_3 \cap M_1| = 8 = |V_1: V_1 \cap M_3|$. Suppose that V_1/Z_1 is not irreducible and let $C_i/Z_i = C_{V_i/Z_i}(\bar{G}_i)$ for $i = 1, 3$. Then, as $|[V_1/Z_1, V_3]| = 2^4$, $C_1 C_3 \leqslant V_1 \cap V_3$. But since $[C_3, T] \leqslant Z_3$, the action of $G_2(2)$ on its natural module implies $C_1 Z_2 = C_2 Z_2$. Hence $C_3 Z_2 \lhd G_2$, which is by (5.5)(b) impossible.

So V_1/Z_1 is the natural $\bar{G}_1$-module. Let $\lambda \in \Delta(\delta_2)$ with $Z_\lambda \leqslant Z_1 Z_3$ but $\delta_1 \neq \lambda \neq \delta_3$. Then, as $V_1 \cap V_3$ is M_1- and M_3-invariant, we obtain

$$A = V_1 \cap V_3 = V_1 \cap V_\lambda = V_3 \cap V_\lambda.$$

Hence $|A/Z_2| = 2$ and A is P-invariant, where P is the parabolic subgroup of G_2 normalizing $\{\delta_1, \delta_3, \lambda\}$. Further $\langle A^{G_2 \cap G_i} \rangle = [V_i, M_2]$ for $i = 1, 3$. Hence if $L_2 = \langle A^{G_2} \rangle$, then L_2/Z_2 is a natural $\bar{G}_2$-module dual to Z_2. (This is a special case of [11], but can also be shown easily the direct way!) As $\Phi(L_2) = 1$, $|L_2 : L_2 \cap V_1| = 2$, but $L_2 \nleqslant M_1$, it follows that $[V_1, L_2]$ is a $G_1 \cap G_2$-invariant 4-group contained in Z_2. But such a group does not exist.

Now $(*)$ implies $[V_1 \cap M_3, V_3 \cap M_1] \leqslant Z_1 \cap Z_3 = 1$ and $[V_1, M_2, V_3] \leqslant Z_1 \cap Z_3 = 1 = [V_3, M_2, V_1]$. Let $L_2 = \langle [V_1, M_2]^{G_2} \rangle$ and $R_2 = \langle V_1^{G_2} \rangle$. Then the double transitivity of G_2 on $\Delta(\delta_2)$ implies $[R_2, L_2] = 1$. Further $[R_2, M_2] = L_2$ and $L_2 \cap V_1 = [V_1, M_2]$. Pick $\delta_4 \in \Delta(\delta_3)$ with $V_3 = (V_3 \cap L_2) Z_4$. Then also $V_3 = (V_3 \cap L_4) Z_2$ and $V_3 \cap L_2 \cap Z_4 = Z_3 = V_3 \cap L_4 \cap Z_2$. ($\delta_2$ and δ_4 correspond to opposite parabolic subgroups of $\bar{G}_3$!) As $Z_2 \leqslant R_4$ we have $L_4 \leqslant M_2$.

Claim

($\dagger$) $V_1 \cap M_3$ centralizes a hyperplane of L_4/Z_4.

If $|L_4 : L_4 \cap M_1| \leqslant 4$, then $|L_4 : Z_4(L_4 \cap M_1)| \leqslant 2$ and as $[V_1 \cap M_3, L_4 \cap M_1] \leqslant Z_1 \cap L_4 = 1$ our claim holds. So we may assume $|L_4 : L_4 \cap M_1| = 8$. Let $B = L_4 \cap M_1 G_1'$. Then $|B : B \cap M_1| = 4$ and B acts quadratically on V_1/Z_1, since all four groups in $U_3(3)$ are conjugate. Hence by (2.16) $|[V_1 \cap M_3/Z_1, B]| = 2$ or 8. In the first case

$$Z_1[V_1 \cap M_3, B] \leqslant Z_1[V_1 \cap M_3, Z_4] \leqslant Z_1 Z_3$$

and thus $[V_1 \cap M_3, B] \leqslant Z_1 Z_3 \cap L_4 = Z_3 \leqslant Z_4$ and ($\dagger$) holds. In the second case

$$Z_2/Z_1 = [V_1/Z_1, Z_4] \leqslant [V_1/Z_1, B] = [V_1 \cap M_3/Z_1, B] \leqslant L_4 Z_1/Z_1$$

by $(*)$ and (2.16), a contradiction to $Z_2 \cap L_4 = Z_3$.

As $V_1 \cap M_3 \nleqslant M_4$ ($\dagger$) shows that L_4/Z_4 is an FF module for $\bar{G}_4$ and $\overline{V_1 \cap M_3}$ induces a GF(2)-transvection on L_4/Z_4. If now $C_3 \neq Z_3$, where $C_3/Z_3 = C_{V_3/Z_3}(\bar{G}_3)$, then $C_3 Z_4 \lhd G_4$, a contradiction to (5.5)(b). So V_3/Z_3 is the natural $\bar{G}_3$-module and $V_3 \cap L_4/Z_4$ is a natural $L_2(2)$-module for $G_3 \cap G_4$. This shows that L_4/Z_4 is a natural $\bar{G}_4$-module dual to Z_4. Hence $L_2 V_1/V_1 \simeq \mathbb{Z}_2$ and thus, if $F_1 = \langle L_2^{G_1} \rangle$, F_1/V_1 either is a trivial $\bar{G}_1$-module or contains a non-trivial composition factor. In the first case $F_1 = V_1 L_2$, a contradiction to

$$Z_1 < [F_1, M_1] \leqslant V_1 \cap L_2.$$

In the second case we obtain a contradiction to $[F_1, V_3] \leqslant [M_2, V_3] \leqslant V_3 \cap L_2$, since $F_1 \leqslant C_{M_1}(Z_2) \leqslant M_2$ as $Z_2 \leqslant V_1 \leqslant R_2 \leqslant C(L_2)$. This proves (5.6).

(5.7). *For each critical pair* (δ_0, δ_d) *we have* $|V_1 : V_1 \cap M_d| = 2 = |V_d : V_d \cap M_1|$ *and* V_1 *resp.* V_d *induce long root involutions on* V_d/Z_d *resp.* V_1/Z_1.

Proof. Suppose (5.7) is false. Then $|V_1 : V_1 \cap M_d| \geqslant 4$ or $|V_d : V_d \cap M_1| \geqslant 4$ and with symmetry we assume the latter. First suppose $|V_1 : V_1 \cap M_d| = 4$. Then $Z_d = [V_1 \cap M_d, V_d] \leqslant V_1$, since V_d cannot centralize a subgroup of index 4 in V_1, and so $Z_{d-1} \leqslant [V_1, V_d] \leqslant V_1$ by the action of $G_2(2)$ on its natural module. Let $W_1 = \langle R_0^{G_1} \rangle$. Then $[W_1, V_1] = 1$ since $d \geqslant 5$ and thus $W_1 \leqslant C_{G_{d-2}}(Z_{d-1}) \leqslant M_{d-1}$. Since $[V_d, V_1, W_1] = 1$, this implies $|[V_d, W_1] : [V_d, V_1]| \leqslant 2$. Hence W_1/V_1 is a trivial module for $H_1 = O^2(\langle V_d^{G_1} \rangle)$. Since H_1 is already transitive on $\Delta(\delta_1)$, this implies $W_1 = R_0$, a contradiction.

Next suppose $|V_1 : V_1 \cap M_d| = 2$. If $Z_2 \leqslant [V_1, V_d]$ we obtain a contradiction as before, arguing with W_d, H_d instead of W_1, H_1. So $Z_1 \nleqslant [V_1, V_d]$ and $[V_1, V_d \cap M_1] = 1$. Since V_d cannot centralize a hyperplane of V_1 we have $Z_d = [V_1 \cap M_d, V_d]$. Further, as $|V_1 : V_1 \cap M_d| = 2$, $|V_d : V_d \cap M_1| = 4$ and by (2.16) $|[V_1, V_d]| = 8$. (I.e. all involutions of $V_d - M_1$ act as long root involutions on V_1/Z_1!) Hence $|[V_1, V_d/Z_d]| = 4$ and $V_d \cap M_1 = C_{V_d}(V_1)$. Pick $\delta_{d+1} \in \Delta(\delta_d)$ with $V_d = (V_d \cap M_1) Z_{d+1}$. Then

$$(*) \quad C_{Z_{d+1}}(V_1) = Z_d = C_{Z_{d+1}}(V_1 \cap M_d)$$

since $Z_{d+1} \cap M_1 = Z_d$ and so $|V_1 \cap M_d : V_1 \cap M_{d+1}| = 4$.

If now (δ_2, δ_{d+2}) is critical for some $\delta_{d+2} \in \Delta(\delta_{d+1})$ then $|Z_{d+1} \cap [V_3, V_{d+2}] Z_d| \geqslant 4$ as above, a contradiction to $(*)$ and $[V_1, V_3] = 1$. So $Z_2 \leqslant M_{d+2}$ for each $\delta_{d+2} \in \Delta(\delta_{d+1})$ and $[Z_2, V_{d+2}] \leqslant Z_{d+2} \cap V_3 = 1$ by $(*)$. Hence $R_{d+1} \leqslant M_2$ and $|R_{d+1} : (R_{d+1} \cap M_1) Z_{d+1}| \leqslant 2$ since $R'_{d+1} = 1$.

But on the other hand, since $V_d/[V_d, M_{d+1}]$ and $[V_d, M_{d+1}]/Z_{d+1}$ are both natural modules for $G_d \cap G_{d+1}$, R_{d+1}/Z_{d+1} contains at least two non-trivial G_{d+1}/M_{d+1}-composition factors. Hence $|[V_1 \cap M_d, H/Z_{d+1}]| \geqslant 4$ for each hyperplane H/Z_{d+1} of R_{d+1}/Z_{d+1}. But this is a contradiction to

$$[V_1 \cap M_d, (R_{d+1} \cap M_1) Z_{d+1}] \leqslant Z_1 Z_{d+1}.$$

So we may with symmetry assume that $|V_1 : V_1 \cap M_d| = 8 = |V_d : V_d \cap M_1|$. Without loss of generality $Z_0 \cap C_{V_1}(V_d) = Z_1$. Pick $\delta_{-1} \in \Delta(\delta_0)$ with $V_{-1} \nleqslant M_{d-2}$. Then $|[V_{-1}, V_{d-2}] \cap Z_0| \geqslant 4$, a contradiction to $V_{d-1} \leqslant C(V_d)$. Thus $V_{-1} \leqslant M_{d-2}$ for each $\delta_{-1} \in \Delta(\delta_0) - \delta_1$ and $[V_{-1}, V_{d-2}] \leqslant$

$Z_{-1}\cap Z_{d-2}=1$. Hence $R_0\leqslant M_{d-1}$ and $R_0\leqslant V_1M_d$. But then $[R_0, V_d]\leqslant V_1\leqslant R_0$, a contradiction to $V_d\nleqslant G_0$.

(5.8). *There exists no group satisfying the hypothesis of this section.*

Proof. Suppose (5.8) is false. Then (5.3)–(5.7) hold and thus $Z_\lambda=[V_1, V_d]=Z_\rho$ for some $\lambda\in\Delta(\delta_1)$, $\rho\in\Delta(\delta_d)$, since $Z_d=[V_1\cap M_d, V_d]$. Now $Z_1\neq Z_d$, since V_d cannot centralize a hyperplane in V_1/Z_1. Hence $[V_1\cap M_d, V_d\cap M_1]=1$ and, since $[V_d, M_\rho]\leqslant V_d\cap M_1$, $[V_1, M_\lambda]\leqslant V_1\cap M_d$, we have

(∗) $[V_d, M_\rho]=C_{V_d}(V_1)$, $[V_1, M_\lambda]=C_{V_1}(V_d)$.

Without loss of generality $Z_0\cap[V_d, M_\rho]=Z_1$. Suppose $V_{-1}\nleqslant M_{d-2}$ for some $\delta_{-1}\in\Delta(\delta_0)-\delta_1$. Then by (5.7) $Z_{-1}\leqslant[V_{-1}, V_{d-2}]$, a contradiction to (∗) since $V_{d-2}\leqslant C(V_d)$. Thus $[V_{-1}, V_{d-2}]\leqslant Z_{-1}\cap Z_{d-2}=1$ for each such δ_{-1} and so $R_0\leqslant M_{d-1}$. Let $V_d^\rho=[V_d, M_\rho]$. Then

$$[V_{-1}, V_d^\rho]\leqslant Z_\rho\cap[V_{-1}, M_0]=Z_\lambda\cap[V_{-1}, M_0].$$

Because of $V_1=[V_1, M_\lambda]Z_0=[V_1, M_0]Z_\lambda$ we have $[M_0, x]\nleqslant Z_0$ for each $x\in Z_\lambda-Z_1$. Hence $[V_{-1}, V_d^\rho]\leqslant Z_1$ and so $V_d^\rho\leqslant M_{-1}$. But then $[V_d^\rho, V_{-1}]\leqslant Z_1\cap Z_{-1}=1$. This implies $R_0\leqslant C(V_d^\rho)$ and thus $R_0\leqslant V_1M_d$ by (2.7). But then

$$[R_0, V_d]=[V_1, V_d]=Z_\lambda\leqslant R_0,$$

a contradiction to $V_d\nleqslant G_0$.

6. Proof of Theorem 3

In this section we carry on with the hypothesis and notation of Section 3. But since we also use the third maximal parabolic subgroup G_2 of the quasiparabolic system $\not{p}$, we write G_{δ_2} for the conjuate of G_0 of distance 2 in Γ.

By Section 5 we know that, if $\bar{G}_0\simeq L_3(2)$ and $|Z_0|=8$, then $\bar{G}_2\simeq\hat{A}_6$ or $\hat{\Sigma}_6$ and Δ: $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!\equiv\!\!\circ$. Hence case (3) of Theorem 3 holds. Thus we may from now on assume that $|Z_0|>8$ if $\bar{G}_0\simeq L_3(2)$. But then by (3.8) and (4.13) $d=1$ or 2. We first treat the case $d=1$.

(6.1). *Suppose $d=1$. Then the following hold*:

(1) $G_0/C_{G_0}(Z_0)\simeq\hat{A}_6$ *or* $\hat{\Sigma}_6$.

(2) $|Z_0|=2^6$ *and is the G_0-module obtained from* $\hat{A}_6\subseteq\mathrm{SL}_3(4)$.

Proof. Let Z_0' be an irreducible $\bar{G}_0$-submodule of Z_0. Then Z_0' is nontrivial. Claim

$(*)$ $Z_0' \nleqslant M_1$.

Suppose $(*)$ is false. Then $Z_0'Z_2' \leqslant M_1$, where Z_2' is the conjugate of Z_0' normal in G_{δ_2}. If $[Z_0', Z_2'] = 1$, then $Z_2' \leqslant M_0$ and thus $Z_0 \leqslant C_{G_1}(Z_2') = M_{\delta_2}$. But then $Z_0 \leqslant M_0 \cap M_2 \leqslant M_1$, a contradiction to $d = 1$.

Thus $Z_2' \nleqslant M_0$ and with symmetry we may assume $|Z_2' : Z_2' \cap M_0| \geqslant |Z_0' : Z_0' \cap M_{\delta_2}|$. So one of the cases of (2.7) holds. But on the other hand $O^{p'p}(P_2) \leqslant G_0 \cap G_1 \cap G_{\delta_2}$, so that $Z_2'M_0 \lhd P_2$. If now $\bar{G}_0 \simeq \hat{A}_6, \hat{\Sigma}_6$ and $|Z_0'| = 2^6$ then $|Z_0' \cap Z_2'| = 2^4$ and thus $Z_2'/Z_2' \cap Z_0'$ is a trivial $O^2(P_2)$-module, a contradiction to $[Z_2', O^2(P_2)] \nleqslant M_0$. If $\bar{G}_0 \simeq U_4(q)$ or $U_5(q)$ and Z_0' is the natural module, then $Z_2'/Z_2' \cap M_0$ is the orthogonal $\bar{P}_2 \simeq L_2(q^2)$-module, while by the action of $P_2 \cap G_{\delta_2}$ on Z_2' it must be the natural module. This together with (2.8)(5) shows that one of the cases (1)–(3) of (2.7) holds. If now $\bar{G}_0 \simeq L_3(q)$ or $\bar{G}_0 \simeq \mathrm{Sp}(4, q)$ with $q > 2$, we obtain a contradiction to $Z_2'M_0/M_0 \simeq Z_2'/Z_2' \cap M_0 \simeq Z_2'/C_{Z_2'}(Z_0')$ as $O^{p'p}(P_2)$-module. (If $\bar{G}_0 \simeq L_3(q)$, then $Z_2'M_0/M_0$ is non-trivial, while $Z_2'/C_{Z_2'}(Z_0')$ is trivial. If $\bar{G}_0 \simeq \mathrm{Sp}(4, q)$ then $|Z_2'M_0/M_0| = q^2$, which is impossible if $q > 2$!) Hence we obtain $|Z_0'| = 2^4$ and $\bar{G}_0 \simeq A_6$ or Σ_6.

Let in this case $V_1' = \langle Z_0'^{G_1} \rangle$, $Z_1' = Z_1 \cap Z_0'$. Then (2.6) implies that V_1'/Z_1' admits no faithfully and quadratically acting 4-group. Hence $|Z_0 : Z_0 \cap M_1| = 2$. Since $[Z_2', Z_0 \cap M_1] \leqslant Z_1'$ and since Σ_6 contains no 4-group, the elements of which act as GF(2)-transvections, this shows that Z_0Z_0' is a trivial $\bar{G}_0$-module. But then $Z_0 = Z_0'$ by definition of Z_0 and $C_{Z_0}(G_0) = 1$, a contradiction. This proves $(*)$.

So $Z_0'/Z_0' \cap M_1$ is a trivial $\bar{P}_2$-module, while Z_1' is a non-trivial $\bar{P}_2$-module. Assume first $\hat{G}_0 = G_0/C_{G_0}(Z_0')$ is a Lie-type group. Then, as $[Z_0', M_1] \leqslant Z_0' \cap M_1$, (2.10)(1), (3.1), and [18] imply $\hat{G}_0 \simeq (P)\mathrm{SL}_3(4)$ or $\mathrm{SL}_3(3)$ and Z_0' is not the natural module. (If $\hat{G}_0 \simeq \mathrm{SL}_3(2)$ then $(*)$ and $Z_0M_1 = Z_0'M_1$ easily imply $Z_0 = Z_0'$ is the natural $L_3(2)$-module, a contradiction to our hypothesis.) Hence Δ: $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!\equiv\!\!\!\equiv\!\!\circ$, since $q = 3$ or 4. Now $\bar{G}_2 \simeq G_2(q)$ or ${}^3D_4(q)$. But since by the structure of $\bar{G}_1$ we have $S = Q_0Z_0'$, this is a contradiction to (2.13).

So $\hat{G}_0 \simeq \hat{A}_6$ or $\hat{\Sigma}_6$ and, since $Z_0'/Z_0' \cap M_1$ is a trivial $\bar{P}_2$-module, $|Z_0'| = 2^6$ by (2.17). (If Z_0' is a 18-dimensional module of (2.17), then $Z_0'/[Z_0', M_1]$ is a non-trivial $\bar{P}_2$-module. Hence, if we consider $\hat{G}_0 \precsim \Gamma L_3(4)$ in its action on Z_0', then a 3-element h of $\hat{P}_2$ is contained in X^∞, where X is the maximal parabolic of $\Gamma L_3(4)$ stabilizing $[Z_0', M_1]$. This shows that h acts non-trivially on $Z_0'/Z_0' \cap M_1$, a contradiction.)

It remains to show $Z_0 = Z_0'$. Suppose this is not the case and let $L_1 = M_1\langle Z_0', Z_2' \rangle$. Then $\delta_0 \sim \delta_2$ in L_1. Assume $|Z_0' : Z_0' \cap M_1| = 4$. Then H contains a 3-element centralizing $Z_0' \cap Z_2'$ and acting faithfully on $Z_0'/Z_0' \cap M_1$, which is impossible by the action of $\hat{\Sigma}_6$ on Z_0'.

So $|Z_0' : Z_0' \cap M_1| = 2$ and $|Z_0' \cap Z_2'| = 2^4$, since $(Z_0' \cap M_2)(Z_2' \cap M_0) \leqslant Z_0' \cap Z_2'$ as $\delta_0 \sim \delta_2$ in L_1. Let $V_1' = \langle (Z_0' \cap M_1)^{L_1} \rangle$. Then, as $|Z_0' \cap M_1 : Z_0' \cap Z_2'| = 2$ and $Z_0 M_1 = Z_0' M_1$ since $Z_0' M_1$ is H-invariant, it follows that $[Z_0, V_1'] \leqslant Z_0'$. But $[Z_0', V_1'] \neq 1$, since $Z_0' \cap M_1 \neq Z_0' \cap Z_2'$, so that Z_0/Z_0' is a trivial G_0-module. This proves (6.1).

(6.2). *Suppose $d = 1$. Then the following holds:*

(1) $G_0 \simeq 2^6 \hat{\Sigma}_6$

(2) $G_2 \simeq 2^{1+6} L_3(2)$

(3) $G_1 = M_1 \cdot (\Sigma_3 \times \Sigma_3)$ *and M_1/Z_1 is the direct sum of a natural $\Omega^+(4, 2)$-module and a natural Σ_3-module.*

Proof. We first show:

(∗) $Z_1 = Z_0 \cap Z_{\delta_2}$ has order 4.

Suppose (∗) is false. Then $|Z_0 \cap Z_{\delta_2}| = 2^4$, since $C_{\bar{G}_1}(M_1) \leqslant M_1$. By the same reason $|Z_0 : Z_0 \cap M_1| = 2$ so that $V_1 > Z_0 \cap Z_{\delta_2}$, where $V_1 = \langle (Z_0 \cap M_1)^{L_1} \rangle$ and $L_1 = M_1 \langle Z_0, Z_{\delta_2} \rangle$. But then $V_1 \nleqslant M_0$, $[Z_0 \cap Z_{\delta_2}, V_1] = 1$, a contradiction, since by $[V_1, O^2(P_2)] \leqslant M_0$, V_1 must act as a field automorphism of $\mathrm{SL}_3(4)$ on Z_0.

Let $K = O^{2'2}(P_2)$, $F_0 = [Z_0, K]$, and $V_1 = \langle F_0^{L_1} \rangle$. Then $|F_0| = 16$ and F_0^l is K-invariant for each $l \in L_1$. Hence the action of $\hat{\Sigma}_6$ on Z_0 implies that $\Phi(V_1) = 1$. As $V_1 \nleqslant M_0$ we have $[Z_0, V_1] = F_0$ and thus $V_1 = F_0 F_2$, where $F_2 = [Z_{\delta_2}, K]$. This implies $L_1/M_1 \simeq \Sigma_3$ and

$$[M_0, V_1] \leqslant [M_0 \cap M_1, V_1][Z_0, V_1] \leqslant Z_1 F_0 = F_0 \leqslant Z_0.$$

This shows that M_0/Z_0 is a trivial G_0'-module and so, since either $Z_0 \leqslant \Phi(M_0)$ or $\Phi(M_0) = 1$, the latter holds. But then $Z_0 = M_0$, as $[M_0, G_0'] \leqslant Z_0$, and $G_0 \simeq 2^6 \cdot \hat{\Sigma}_6$ or $2^6 \hat{A}_6$ by the Frattini argument. Now the second possibility contradicts (∗). So (1) holds.

Let $W_1 = \langle (Z_0 \cap M_1)^{L_1} \rangle$. Then, as V_1/Z_1 is the natural $\Omega^+(4, 2)$-module and $[Z_0 \cap M_1, Z_{\delta_2} \cap M_1] \leqslant Z_1$, $|S| = 2^{10}$ implies $|W_1 : V_1| = 4$, $W_1' = Z_1$, and $W_1 = M_1$. Now (3) is easy to see. Moreover, since $|S/Z| = 2^9$ and M_2/Z is a non-trivial module for $\bar{G}_2$ we have $\bar{G}_2 \in \{L_3(2), A_6, \Sigma_6\}$. The action of $\hat{\Sigma}_6$ on Z_0 implies $|[M_2 \cap G_0', M_0]| = 4$. Hence if $M_2 \leqslant G_0'$ then $|M_2/Z| \leqslant 2^5$ and M_0 induces GF(2)-transvections on M_2/Z, which is impossible since Q_1 contains three non-trivial $\bar{P}_1$-sections.

So $M_2 \nleqslant G_0'$, $|[M_0, M_2]| = 2^4$ and $|M_2| = 2^7$. Since $M_0 M_2 = Q_1$ and since A_6 has no non-trivial GF(2)-module V with two non-trivial $\bar{P}_1$-sections and

$|[H, \bar{Q}_1]| = 2$ for some hyperplane H of V (as $|[M_2 \cap G_0'/Z, M_0]| = 2!$) we obtain $\bar{G}_2 \simeq L_3(2)$. Since M_2/Z contains two non-trivial $\bar{P}_1$-sections this shows (2).

(6.3). *Suppose $d = 2$. Then $|Z_0| = 2^4$ and $\hat{G}_0 = G_0/C_{G_0}(Z_0) \in \{A_6, \Sigma_6\}$.*

Proof. Without loss $|Z_{\delta_2}: Z_{\delta_2} \cap M_0| \geqslant |Z_0: C_{Z_0}(Z_{\delta_2})|$. Hence one of the cases of (2.7) holds for Z_0 and $\hat{G}_0$. Further $O^{p'p}(P_2) \leqslant G_0 \cap G_1 \cap G_{\delta_2}$ and thus acts on $Z_{\delta_2}M_0/M_0$. Hence exactly the same argument as in the first part of the proof of (6.1) proves (6.3).

(6.4). *Suppose $d = 2$. Then the following hold*:

(1) $G_0 = M_0 R_0$, $R_0 \simeq \hat{A}_6$, $Z_0 = \Phi(M_0)$ *is the natural* Sp(4, 2)′*-module, while* $|M_0/Z_0| = 2^6$ *and is the module obtained from* $\hat{A}_6 \subseteq \mathrm{SL}_3(4)$.

(2) $G_1 = M_1 R_1$, $R_1 \simeq \Sigma_3 \times A_5$, $\Phi(M_1) = Z_1$, *and* $|M_1/Z_1| = 2^8$ *and is the direct sum of two orthogonal* $A_5 \simeq \Omega^-(4, 2)$*-modules, which are permuted by* Σ_3.

(3) $G_2 \simeq 2^{1+6}\Omega^-(6, 2)$ *non-split.*

Proof. By (6.3) we have

$$Z_1 = Z_0 \cap Z_{\delta_2} = Z_0 \cap M_{\delta_2} = Z_{\delta_2} \cap M_0.$$

If now $\Phi(M_0) = 1$, then $|M_0: Z_0| \leqslant 2$ since $|H^1(\hat{G}_i, Z_0)| = 2$ and $Z_0 Z_{\delta_2} \lhd L_1 = M_1\langle M_0, M_2\rangle$. But this is impossible since L_1 is already transitive on $\Delta(\delta_1)$ and since all involutions of $Z_0 Z_{\delta_2}$ are contained in $Z_0 \cup Z_{\delta_2}$.

Thus $Z_0 \leqslant \Phi(M_0)$. As $[Z_{\delta_2}, M_0 \cap M_1] \leqslant Z_1 \leqslant Z_0$ we have $|M_0: (M_0 \cap M_1)\Phi(M_0)| \geqslant 4$, since otherwise $O^2(\langle Z_{\delta_2}^{G_0}\rangle)$ would centralize $M_0/\Phi(M_0)$. Hence $|M_0 M_1: M_1 \Phi(M_0)| \geqslant 4$. On the other hand, since $\not{p}$ is a parabolic or quasiparabolic system, we have $\bar{P}_0 \in \{\Sigma_3, \mathrm{Sz}(2), L_2(4), L_2(8)\}$. Hence $\bar{P}_0 \simeq L_1/M_1 \simeq L_2(4)$ or $L_2(8)$. In the second case $|M_0/M_0 \cap M_1| = 8$ and $H \cap \tilde{P}_0 \simeq \mathbb{Z}_7$ acts faithfully on this group. Since $\bar{G}_0'$ is generated by three conjugates of $\bar{Z}_{\delta_2}$ this would imply that $A_6 \times \mathbb{Z}_7$ or $\hat{A}_6 \times \mathbb{Z}_7$ acts faithfully on an elementary group of order smaller or equal to 2^9, which is impossible.

Thus $\bar{P}_0 \simeq L_2(4)$, $\bar{G}_1 \simeq \Sigma_3 \times A_5$, and obviously $\bar{G}_2 \simeq \mathrm{PSU}_4(2)$. Further by (2.6) V_1/Z_1 is the direct sum of two Steinberg modules for L_1/M_1. Let H_0 be generated by three conjugates of Z_{δ_2} such that $G_0' \leqslant M_0 H_0$ and $\hat{M}_0 = M_0/Z_0$. Then $|\hat{M}_0: C_{\hat{M}_0}(H_0)| \leqslant 2^6$. On the other hand, since $O^{2'2}(P_2)$ centralizes $M_0/M_0 \cap M_1$ it is easy to see that $\hat{M}_0/C_{\hat{M}_0}(H_0)$ does not contain a natural $H_0/H_0 \cap M_0 \simeq \mathrm{Sp}(4, 2)'$-module. Hence $|\hat{M}_0: C_{\hat{M}_0}(H_0)| = 2^6$ and $H_0/H_0 \cap M_0 \simeq \hat{A}_6$. Further, since $[S, H] \leqslant Q_2$, $\bar{G}_0 \simeq \hat{A}_6$. Let C be the co-image of $C_{\bar{M}_0}(H_0)$. Then $\hat{C}_0 = C_{\hat{C}}(M_0) \neq 1$ if $C > Z_0$. But $C_0 \leqslant Z(M_0)$, since Z_0 and M_0/C are non-equivalent $\bar{G}_0$-modules. Hence easily $\Omega_1(C_0) > Z_0$, a contradiction to $Z < Z_0$ and $\bar{G}_0 \simeq \hat{A}_6$. (An extension of a natural Sp(4, 2)-module by a trivial splits over A_6!)

This shows $C = Z_0 = \Phi(M_0)$ and $V_1 = M_1$. Now the Frattini argument implies (1) and (2). Since $|M_2/Z| = 2^6$ and is acted upon by $\bar{G}_2$ it is easy to see that M_2 is extraspecial. Now (3) follows by a well known result of R. L. Griess [21].

Since we may assume that $d = 1$ or 2 to prove Theorem 3, as shown in the introduction to this section, (6.2) and (6.4) toghether prove Theorem 3.

7. Proof of Theorems 1 and 2

Since the proof of Theorem 2 relies heavily on the results of [14], we state these results here for the convenience of the reader.

Hypothesis A. Let p be a prime and G be a group which is generated by three finite subgroups P_1, P_2, P_3 such that for $B = P_1 \cap P_2 \cap P_3$, $S \in \mathrm{Syl}_p(B)$, and $G_\alpha = \langle P_1, P_2 \rangle$, $G_\beta = \langle P_1, P_3 \rangle$ the following hold:

(a) $O^{p'}(P_i/O_p(P_i)) \in \{L_2(p^{n_i}),\ \mathrm{SL}_2(p^{n_i}),\ U_3(p^{n_i}),\ \mathrm{SU}_3(p^{n_i}),\ \mathrm{Sz}(2^{n_i})$ (and $p = 2$), $\mathrm{Ree}(3^{n_i})$ (and $p = 3$)$\}$

(b) $B = N_{P_i}(S)$, $i = 1, 2, 3$

(c) $Z = \Omega_1(Z(S))$ is normal neither in G_α nor in G_β

(d) $O_p(P_1) \cap O_p(P_j)$ is normal neither in P_1 nor in P_j for $j = 2, 3$

(e) No non-trivial subgroup of B is normal in G.

If now $\not{p}$ is a quasiparabolic system of rank 3 with connected diagram Δ of some group G with $B_G = 1$ (B defined as in (2.1)!), then it has been shown in [19, (2.3)] that G satisfies, with suitably changed notation, (a), (b), (d) and (e) of Hypothesis A. (I.e., the connectedness of Δ implies (d)!) So either Hypothesis A holds or Z is normal in G_α or G_β. In the latter case we will, to prove Theorem 2, apply Theorem 3.

(7.1). *Suppose now that G is a group satisfying Hypothesis* A. *For $\lambda \in \{\alpha, \beta\}$ let*

$$Z_\lambda = \langle Z^{G_\lambda} \rangle$$

$$Q_\lambda = S_{G_\lambda}.$$

L_λ is the smallest normal subgroup of G_λ containing $O^{p'}(P_1)$.

$\tilde{Z}_\lambda = Z_\lambda / C_{Z_\lambda}(L_\lambda)$

$C_\lambda = C_{L_\lambda}(\tilde{Z}_\lambda)$

$\bar{G}_\lambda = G_\lambda / C_\lambda$ *(Here we consider* $^{-}$ *as natural homomorphism.)*

Let $\Gamma = \Gamma(G_\alpha, G_\beta)$ *be the coset graph of* G_α, G_β *in* G *and* d *be the same parameter for* Γ *as defined in Section* 3. *Then by Theorem* 1 *of* [14], *up to interchanging* α *and* β, *one of the following holds*:

(I) $d=2$, $\bar{L}_\lambda \simeq \mathrm{SL}_3(3)$, *and* Z_λ *is the natural* $\bar{L}_\lambda$*-module,* $Q'_\lambda = Z_\lambda$ *and* Q_λ/Z_λ *is the natural* $\bar{L}_\lambda$*-module dual to* Z_λ *for* $\lambda \in \{\alpha, \beta\}$.

(II) $d=1$, $|Z_\beta : Z_\beta \cap Q_\alpha| > |Z_\alpha : Z_\alpha \cap Q_\beta|$, $Q_\beta = Z_\beta$, $Z_\beta \cap Z(L_\beta) = 1$, *and one of the following holds*:

(a) $\bar{L}_\beta \simeq \Omega^-(6, q)$, $q = p^m$, *and* Z_β *is the natural* $\bar{L}_\beta$*-module.* $\bar{L}_\alpha \simeq \mathrm{SL}_3(q)$, Z_α *is the natural* $\bar{L}_\alpha$*-module,* $\Phi(Q_\alpha) = Z_\alpha$, *and* Q_α/Z_α *is the direct sum of two natural* $\bar{L}_\alpha$*-modules dual to* Z_α.

(b) p *is odd,* $\bar{L}_\beta \simeq \Omega(5, q)$, $q = p^m$, *and* Z_β *is the natural* $\bar{L}_\beta$*-module.* $\bar{L}_\alpha \simeq \mathrm{SL}_3(q)$, Z_α *is the natural* $\bar{L}_\alpha$*-module,* $\Phi(Q_\alpha) = Z_\alpha$, *and* Q_α/Z_α *is the natural* $\bar{L}_\alpha$*-module dual to* Z_α.

(c) $p=2$, $\bar{L}_\alpha \simeq \mathrm{Sp}(4, q)$, $Z_\alpha = Q_\alpha$, $\tilde{Z}_\alpha$ *is the natural* $\bar{L}_\alpha$*-module, and* $|Z_\alpha \cap Z(L_\alpha)| = q$. *There exists an* $N \lhd L_\beta$, $N < Z_\beta$, *such that* $L_\beta / C_{L_\beta}(N) \simeq \mathrm{SL}_3(q) \simeq L_\beta / C_{L_\beta}(Z_\beta/N)$ *and* N *and* Z_β/N *are natural* $\mathrm{SL}_3(q)$*-modules dual to each other.*

(III) $d=1$, $|Z_\alpha : Z_\alpha \cap Q_\beta| = |Z_\beta : Z_\beta \cap Q_\alpha|$, $\Phi(Q_\alpha) = \Phi(Q_\beta) = 1$, $\bar{L}_\delta \in \{\mathrm{Sp}(4,2)', \mathrm{Sp}(4,2), \mathrm{Sp}(4,4), G_2(2), \mathrm{SL}_3(q)\}$ *for* $\delta \in \{\alpha, \beta\}$ *and one of the following holds*:

(a) $\bar{L}_\delta \simeq A_6$ *for* $\delta = \alpha$ *and* β. $[Q_\delta, L_\delta] = Q_\delta$ *is the natural* $\bar{L}_\delta$*-module and* S *splits over* Q_δ.

(b) $\bar{L}_\alpha \simeq A_6$, $\bar{L}_\beta \simeq \mathrm{Sp}(4, 2)$, $[Q_\delta, L_\delta]$ *is the natural* $\bar{L}_\delta$*-module for* $\delta \in \{\alpha, \beta\}$, *and* S *splits over* Q_δ.

(c) $\bar{L}_\delta \simeq \mathrm{Sp}(4, 2)$, $[Q_\delta, L_\delta]$ *is the natural* $\bar{L}_\delta$*-module for* $\delta \in \{\alpha, \beta\}$, *and* $|Z_\alpha : Z_\alpha \cap Q_\beta| = 2$.

(d) $\bar{L}_\delta \simeq \mathrm{Sp}(4, 2)$, $[Q_\delta, L_\delta]$ *is the natural* $\bar{L}_\delta$*-module for* $\delta \in \{\alpha, \beta\}$, *and* $|Z_\alpha : Z_\alpha \cap Q_\beta| = 4$.

(e) $\bar{L}_\delta \simeq \mathrm{Sp}(4, 4)$, $Q_\delta = [Q_\delta, L_\delta]$ *is the natural* $\bar{L}_\delta$*-module, and* S *does not split over* Q_δ *for* $\delta \in \{\alpha, \beta\}$.

(f) $\bar{L}_\delta \simeq \mathrm{SL}_3(2)$, $Q_\delta = Z_\delta$, $[Q_\delta, L_\delta]$ *is the natural* $\mathrm{SL}_3(2)$*-module, and* $|Z_\delta \cap Z(L_\delta)| = 2$ *for* $\delta \in \{\alpha, \beta\}$.

(g) $\bar{L}_\delta \simeq \mathrm{SL}_3(q)$ *and* $Z_\delta = Q_\delta = [Q_\delta, L_\delta]$ *is the naturl* $\bar{L}_\delta$*-module for* $\delta \in \{\alpha, \beta\}$.

(h) $\bar{L}_\delta \simeq G_2(2)$, $Q_\delta = Z_\delta = [Q_\delta, L_\delta]$ *is the natural* $\bar{L}_\delta$*-module, and* S *does not split over* Q_δ *for* $\delta \in \{\alpha, \beta\}$.

(IV) $Z \nleq Q_\alpha$ *and* $Z \nleq Q_\beta$ *and one of the following holds*:

(a) $Q_\alpha = Q_\beta = 1$

(b) $Q_\alpha \simeq Q_\beta \simeq \mathbb{Z}_2$ *and* L_α/Q_α *and* L_β/Q_β *are parabolic isomorphic to one of the following groups*: $\mathrm{Sp}(4, 2)$, $U_4(2)$, $U_5(2)$, $G_2(2)$, $G_2(2)'$, ${}^2F_4(2)$, ${}^2F_4(2)'$, ${}^3D_4(2)$, J_2, M_{12}, *or* $\mathrm{Aut}(M_{12})$. *Moreover,* Z *is normal in* P_1.

(c) $Q_\alpha = 1$, $|Q_\beta| = q$, $q = 2$ *or* 4, *and* L_α *resp.* L_β/Q_β *are parabolic isomorphic to* $\mathrm{Sp}(4, q)$ *resp.* $L_3(q)$. *Further, if* $q = 4$, *then* S *does not split over* Q_β.

(d) $Q_\alpha \simeq Q_\beta \simeq \mathbb{Z}_4$ *and* L_δ/Q_δ *is parabolic isomorphic to* $G_2(2)$ *or* $G_2(2)'$ *for* $\delta \in \{\alpha, \beta\}$.

(e) $Q_\alpha \simeq Q_\beta \simeq \mathbb{Z}_3$, L_δ/Q_δ *is parabolic isomorphic to* $P\,\mathrm{Sp}(4, 3)$ *or* $U_4(3)$ *for* $\delta \in \{\alpha, \beta\}$, *and in the latter case* S *does not split over* Q_δ.

(V) $Z \nleq Q_\alpha$, $Z \leqslant Q_\beta$, $Q'_\beta = 1$, $\bar{L}_\beta \simeq \mathrm{SL}_3(q)$, $[Z_\beta, L_\beta]$ *is the natural* $\bar{L}_\beta$*-module, and* S *does not split over* Q_β. *Moreover, one of the following hold*:

(a) $q = 2$ *or* 5. $Q_\alpha = 1$, $Q_\beta = [Z_\beta, L_\beta]$, *and* L_α *is parabolic isomorphic to* $G_2(q)$.

(b) $q = 2$, $Q_\alpha = 1$, $C_{Q_\beta}(L_\beta) = 1$, *and* L_α *is parabolic isomorphic to* M_{12}.

(c) $q = 2$, $Q_\alpha \simeq C_{Q_\beta}(L_\beta) \simeq \mathbb{Z}_2$, *and* L_α/Q_α *is parabolic isomorphic to* M_{12}.

(d) $q = 2$, $Q_\alpha \simeq C_{Q_\beta}(L_\beta) \simeq \mathbb{Z}_2$, *and* L_α/Q_α *is parabolic isomorphic to* $\mathrm{Aut}(M_{12})$.

(7.2) *Proof of Theorem* 2. Suppose that $\not{p} = \{P_0, P_1, P_2\}$ is a quasiparabolic system of the group G, satisfying the hpothesis but not the conclusion of Theorem 2. Then by the main theorem of [19] $\Delta(I)$ is linear and, since case (1) of Theorem 2 does not hold, [16, (3.2)] shows that $\Delta(I)$ is not spherical. Using the notation and reduction of (2.1) we have without loss of generality

(1) $G = G_0 = \langle \tilde{P}_i \mid i \in I \rangle$, $I = \{0, 1, 2\}$, $P_i = \tilde{P}_i H$, and $H = H_0 H_1 H_2$

(2) $B_G = 1$, since we may assume $S_G = 1$ and since then $B_G \leqslant H \cap Z(G)$.

Let $Z = \Omega_1(Z(S))$ and choose an enumeration of Δ so that G_0, G_2 are the "connected" maximal parabolics. Suppose first:

$(*)$ $C_{\bar{G}_i}(M_i) \leqslant M_i$ for $i = 0, 2$.

If in addition $Z \ntriangleleft G_i$ for $i=0, 2$, then Hypothesis A holds and G satisfies one of I, II, III of (7.1) with $\{L_\alpha, L_\beta\} = \{\tilde{G}_0, \tilde{G}_2\}$. In case I Δ is spherical. If in case II $\Delta(I)$ is not spherical, then $\tilde{G}_i/M_i$ must be parabolic isomorphic but not isomorphic to one of the Lie-type groups in (7.1)II for $i=0$ or 2. Hence by definition of quasiparabolic systems, $\tilde{G}_i/M_i \simeq \hat{\Sigma}_6$ and II(c) holds. But then it is obvious that case (7) of Theorem 2 holds.

So G satisfies one of the cases of (7.1)III. Hence to show that case (4) of Theorem 2 holds, it suffices to show that III(e) and (h) are not satisfied (since $\Delta(I)$ is not spherical). Suppose first III(h) holds. Then $\{M_0, M_2\} \subseteq \mathcal{O}(S)$. Now in the action of $G_2(2)$ on its natural module a 2-Sylow subgroup of $G_2(2)$ contains exactly three offending subgroups. Futher, one parabolic normalizes one of these, while the other acts as Σ_3 on these. Hence

$$\mathcal{O}(S) = M_0 \cup \{M_2^{P_2}\} = M_2 \cup \{M_0^{P_0}\} \qquad \text{and} \qquad |\mathcal{O}(S)| = 4.$$

But this is a contradiction to the fact that G_1 acts doubly transitive on $\mathcal{O}(S)$ and $G_1/M_1 \simeq \Sigma_3 \times \Sigma_3$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3)\mathbb{Z}_2$.

In III(e) we obtain a contradiction to (2.2). So we may assume $Z \lhd G_i$ for $i=0$ or 2 if $(*)$ holds. But then it suffices by Theorem 3 to show that $C_{\tilde{G}_1}(M_1) \leqslant M_1$. If this is not the case, then obviously

$$\tilde{G}_1 = M_1 C_{\tilde{G}_1}(M_1) \tilde{P}_i \qquad \text{for} \quad i=0, 2.$$

Hence $1 \neq Z \cap M_1 \lhd G = \langle \tilde{G}_1, \tilde{G}_i \rangle$, a contradiction.

So we may assume $C_{\tilde{G}_0}(M_0) \nleqslant M_0$. We show that in this non-constrained case again Hypothesis A holds. Suppose this is not the case. Assume first $Z \ntriangleleft G_2$. Then $Z \lhd G_0$. If $C_{\tilde{G}_2}(M_2) \nleqslant M_2$, then $\tilde{G}_2 = C_{\tilde{G}_2}(M_2)S$ and so $M_2 = 1$, since otherwise $1 \neq Z \cap M_2 \lhd \langle \tilde{G}_0, \tilde{G}_2 \rangle = G$. If now $M_0 \leqslant Z(\tilde{G}_0)$, then $M_0 \cap O^p(\tilde{G}_0) \neq 1$ since $Z \lhd G_0$. Hence the hypothesis of (2.21)(2) holds for $R = O^p(\tilde{G}_0)$ and $P = P_1 \cap O^p(\tilde{G}_0)$, a contradiction since Δ is not spherical. So $M_0 \nleqslant Z(\tilde{G}_0)$ and thus $M_0 \nleqslant Z(P_1 \cap \tilde{G}_0)$. Hence, applying the description of parabolic subgroups of elements of $\mathscr{L}_p^2$ in [17, (3.2)] to P_1, (2.2) implies $\tilde{G}_2 \simeq {}^2F_4(2)$ or ${}^2F_4(2)'$. But this is impossible, since $\bar{P}_1 \simeq \mathrm{Sz}(2)$ as $P_1 \leqslant N(Z)$.

So $C_{\tilde{G}_2}(M_2) \leqslant M_2$. Hence $\tilde{P}_1$ has two non-central chief factors in Q_1. On the other hand $M_0 \cap O^p(\tilde{G}_0) \neq 1 = [M_0, O^p(\tilde{G}_0)]$, so that (2.21)(1) applied to $O^p(\tilde{G}_0)$ supplies a contradiction.

This shows $Z \lhd G_2$ and thus, as above, $M_0 = 1$. If $C_{\tilde{G}_2}(M_2) \nleqslant M_2$, we get a contradiction as above by reversing the roles of G_0, G_2. Hence $P_1 \leqslant N(Z)$ and P_1 has two non-central chief factors in Q_1. Now again [17, (3.2)] implies $\tilde{G}_0 \simeq {}^2F_4(q)$ or ${}^2F_4(2)$, whence $\bar{P}_1 \simeq \mathrm{Sz}(q)$, a contradiction as before.

So again Hypothesis A holds and thus we have one of the cases of (7.1) IV or V. We show that in each case Theorem 2 holds. First assume IV holds. If $M_0 = 1 = M_2$, then by [17, (3.2)] $\tilde{G}_0 \simeq \tilde{G}_2$ and G_1 is solvable, since $[Q_i, \tilde{P}_i] \leqslant M_1$ for $i = 0, 2$. Hence case (2) or (8) of Theorem 2 holds, a contradiction. In IV(c) Δ would be spherical, a contradiction.

Suppose next case IV(e) holds. If $\tilde{G}/M_0 \simeq \tilde{G}_2/M_0 \simeq U_4(3)$, then P_1 is solvable, since otherwise $Z(\tilde{G}_0) = Z(\tilde{P}_1) = Z(\tilde{G}_2) \simeq \mathbb{Z}_3$. Hence $\tilde{G}_1/M_1 \simeq L_2(9) \times L_2(9)$ and $|Q_i : M_1| = 9$ for $i = 0, 2$. But this is impossible by the structure of a parabolic of $U_4(3)$. So $\tilde{G}_0 \simeq \tilde{G}_2 \simeq \mathbb{Z}_3 \times P\mathrm{Sp}(4, 3)$ in (e). Since $Z(\tilde{G}_0) \neq Z(\tilde{P}_1)$ we have $Z(\tilde{P}_1) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. If now $Z(G_0) = Z(\tilde{G}_0)$, then $C_{M_1}(P_0) = C_{M_1}(P_2) = Z(G_0)$, a contradiction. Thus by symmetry $G_0 \simeq (\mathbb{Z}_3 \times P\,\mathrm{Sp}(4, 3))\mathbb{Z}_2 \simeq G_2$ and case (9) of Theorem 2 holds, a contradiction.

Next we show that IV(b) is impossible. Namely if (b) holds for $\tilde{G}_0$, $\tilde{G}_2$, then by the structure of P_i, $i = 0, 2$,

$$V_1 = \langle Z^{G_1} \rangle = \langle Z^{P_i} \rangle, \qquad i = 0, 2$$

has order 8 resp. 2^5 in the case of $U_4(2)$, $U_5(2)$. Now the latter case is impossible, since we would have $\bar{G}_1 \simeq A_5 \times A_5$ and $|Q_i : M_1| = 4$ for $i = 0, 2$. Thus in any case $G_1 = P_i C_{G_1}(V_1)$ for $i = 0, 2$, since $G_1/C_{G_1}(V_1) \lesssim L_3(2)$. Hence $C_{V_1}(G_1) = C_Z(P_i)$, $i = 0, 2$ such that $C_{V_1}(G_1) \lhd G = \langle \tilde{G}_0, \tilde{G}_1 \rangle$, a contradiction.

Finally in case IV(d) $P_i \nleqslant N(Z)$ for $i = 0, 2$, since $G = \langle G_i, P_i \rangle$. Since again $|Z| = 4$, we obtain also in this case $\langle Z^{G_1} \rangle = \langle Z^{P_i} \rangle$, $i = 0, 2$ has order 8, a contradiction as before.

So finally case (7.1)V remains to be treated. Now in V by definition of a quasiparabolic system only V(a) is possible. If now $p = 5$ then Theorem 2 of [17] implies that $\not{p}$ is not a parabolic system. (Together with Section 4 of [15] in which it was shown that one obtains from a parabolic system a classical Tits chamber system of the same type!) If $p = 2$ it is an easy exercise to show that case (3) of Theorem 2 holds. This finishes the proof of Theorem 2.

(7.3) *Proof of Theorem* 1. Suppose that the hypothesis of Theorem 1 is satisfied. Then either one of the cases (2)–(5) of Theorem 2 of [17] holds or there is a parabolic system of he same type as described in case 1 of that theorem. In (2)–(4) it is clear that case (2), (7), or (4) with $p = 5$ of Theorem 1 holds. Case (5) of [17] is exactly case (9) of Theorem 1. Moreover, if Δ is spherical, then case (1) holds by [16, (3.2)]. So it finally remains to be shown that Theorem 1 is a consequence of Theorem 2, if there is a parabolic system of the same type corresponding to $\mathscr{C}$. (As stated in case (1) of [17, Theorem 2].) But this is an easy exercise, which is left to the reader.

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